SCHWARZSCHILD METRIC - THE NEWTONIAN LIMIT & CHRISTOFFEL SYMBOL WORKSHEET

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In our derivation of the Schwarzschild metric, we got as far as finding the dependence of the metric on the spacetime coordinates, giving the form

$$ds^2 = -\left(1+\frac{X}{r}\right)dt^2 + \left(1+\frac{X}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1}$$

The final task is to find the constant X, which we can do by considering the behaviour of the metric for large r and requiring that it reduce to the Newtonian graviational force law in that limit. [I've renamed the constant C in the original post to X here to avoid confusion with the C that turns up in the metric tensor below.]

For an object initially at rest, the spatial comopnents of its four-velocity are all zero: $u^i = 0$. However, the contraction of $\mathbf{u}$ with itself gives the invariant $\mathbf{u}\cdot\mathbf{u} = -1$, so we have

$$\mathbf{u}\cdot\mathbf{u} = g_{\mu\nu}u^\mu u^\nu \tag{2}$$

$$= g_{tt}\left(u^t\right)^2 = -1 \tag{3}$$

$$u^t = \sqrt{-\frac{1}{g_{tt}}} \tag{4}$$

Any object's trajectory obeys the geodesic equation which, in terms of Christoffel symbols, is

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0 \tag{5}$$

where a dot denotes a derivative with respect to proper time τ, so that $\dot{x}^\nu = u^\nu$.

In our case, this reduces to

$$\ddot{x}^{\mu}+\Gamma_{tt}^{\mu}\left(u^{t}\right)^{2}=\ddot{x}^{\mu}-\frac{\Gamma_{tt}^{\mu}}{g_{tt}}=0 \tag{6}$$

$$\ddot{x}^{\mu}=\frac{\Gamma_{tt}^{\mu}}{g_{tt}}=-\frac{1}{A}\Gamma_{tt}^{\mu} \tag{7}$$

where $A=-g_{tt}$. We therefore need to calculate the Christoffel symbols Γ_{tt}^{μ}, which we can do from their expression in terms of $g_{\mu\nu}$:

$$\Gamma_{\nu\sigma}^{\mu}=\frac{1}{2}g^{\mu\lambda}\left(\partial_{\sigma}g_{\nu\lambda}+\partial_{\nu}g_{\lambda\sigma}-\partial_{\lambda}g_{\sigma\nu}\right) \tag{8}$$

This can get quite tedious, but Moore provides a worksheet in the Appendix which simplifies the task. The notation is the same as that used for Ricci tensor worksheet for the generic diagonal metric, written as

$$ds^{2}=-A\left(dx^{0}\right)^{2}+B\left(dx^{1}\right)^{2}+C\left(dx^{2}\right)^{2}+D\left(dx^{3}\right)^{2} \tag{9}$$

where x^0 is the time coordinate and the other three are space coordinates. Note the minus sign in the first term: this makes explicit the fact that the metric component for time should be negative. Thus we have $g_{00}=-A$, $g_{11}=B$, $g_{22}=C$ and $g_{33}=D$.

Derivatives with respect to coordinates are written as subscripts, so that $A_{01}=\frac{\partial^2 A}{\partial x^0 \partial x^1}$ and so on. It's important not to confuse this notation with tensor notation; A_{01} is *not* the 01 component of a tensor. Although we don't need all the Christoffel symbols here, I've produced the table for reference.

$\Gamma_{00}^{0}=\frac{1}{2A}A_{0}$	$\Gamma_{10}^{0}=\Gamma_{01}^{0}=\frac{1}{2A}A_{1}$	$\Gamma_{20}^{0}=\Gamma_{02}^{0}=\frac{1}{2A}A_{2}$	$\Gamma_{30}^{0}=\Gamma_{03}^{0}=\frac{1}{2A}A_{3}$
$\Gamma_{11}^{0}=\frac{1}{2A}B_{0}$	$\Gamma_{22}^{0}=\frac{1}{2A}C_{0}$	$\Gamma_{33}^{0}=\frac{1}{2A}D_{0}$	other $\Gamma_{\mu\nu}^{0}=0$
$\Gamma_{01}^{1}=\Gamma_{10}^{1}=\frac{1}{2B}B_{0}$	$\Gamma_{11}^{1}=\frac{1}{2B}B_{1}$	$\Gamma_{12}^{1}=\Gamma_{21}^{1}=\frac{1}{2B}B_{2}$	$\Gamma_{13}^{1}=\Gamma_{31}^{1}=\frac{1}{2B}B_{3}$
$\Gamma_{00}^{1}=\frac{1}{2B}A_{1}$	$\Gamma_{22}^{1}=-\frac{1}{2B}C_{1}$	$\Gamma_{33}^{1}=-\frac{1}{2B}D_{1}$	other $\Gamma_{\mu\nu}^{1}=0$
$\Gamma_{02}^{2}=\Gamma_{20}^{2}=\frac{1}{2C}C_{0}$	$\Gamma_{12}^{2}=\Gamma_{21}^{2}=\frac{1}{2C}C_{1}$	$\Gamma_{22}^{2}=\frac{1}{2C}C_{2}$	$\Gamma_{32}^{2}=\Gamma_{23}^{2}=\frac{1}{2C}C_{3}$
$\Gamma_{00}^{2}=\frac{1}{2C}A_{2}$	$\Gamma_{11}^{2}=-\frac{1}{2C}B_{2}$	$\Gamma_{33}^{2}=-\frac{1}{2C}D_{2}$	other $\Gamma_{\mu\nu}^{2}=0$
$\Gamma_{03}^{3}=\Gamma_{30}^{3}=\frac{1}{2D}D_{0}$	$\Gamma_{13}^{3}=\Gamma_{31}^{3}=\frac{1}{2D}D_{1}$	$\Gamma_{23}^{3}=\Gamma_{32}^{3}=\frac{1}{2D}D_{2}$	$\Gamma_{33}^{3}=\frac{1}{2D}D_{3}$
$\Gamma_{00}^{3}=\frac{1}{2D}A_{3}$	$\Gamma_{11}^{3}=-\frac{1}{2D}B_{3}$	$\Gamma_{22}^{3}=-\frac{1}{2D}C_{3}$	other $\Gamma_{\mu\nu}^{3}=0$

In our case, we need only the Γ^{μ}_{00} terms, which occur in the first column. Since $A = \left(1+\frac{X}{r}\right)$ the derivatives with respect to t, θ and ϕ are all zero, and the only non-zero Christoffel symbol is

$$\Gamma^{r}_{tt} = \Gamma^{1}_{00} = \frac{1}{2B}A_1 = \frac{1}{2}\left(1+\frac{X}{r}\right)\frac{\partial A}{\partial r} = -\frac{X}{2r^2}\left(1+\frac{X}{r}\right) \tag{10}$$

Therefore from 7 we have

$$\ddot{x}^r = \frac{d^2 r}{d\tau^2} = -\frac{1}{A}\left(-\frac{X}{2r^2}\left(1+\frac{X}{r}\right)\right) = \frac{X}{2r^2} \tag{11}$$

For large r, the Schwarzschild metric reduces to flat space, so the radial coordinate becomes the Newtonian radial coordinate and the proper time τ becomes the Newtonian time t, so

$$\frac{d^2 r}{d\tau^2} \to \frac{d^2 r}{dt^2} = \frac{X}{2r^2} \tag{12}$$

This is equivalent to Newton's law of gravity for a mass a distance r from a mass M if

$$X = -2GM \tag{13}$$

[The minus sign indicates that the test mass accelerates towards M, that is, in the direction of decreasing r.]

Making this substitution in 1 we get the final form of the Schwarzschild metric

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{14}$$

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