SCHWARZSCHILD METRIC - TIME COORDINATE

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The time component of the Schwarzschild metric does not correspond to proper time, even for an object at rest. This metric is, for a spherical mass M:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1}$$

For an object at rest, the invariant interval is given by the proper time. That is, we assume that the relation from special relativity applies in curved space-time as well:

$$ds^2 = -d\tau^2 \tag{2}$$

For the Schwarzschild metric, this means that for an object at rest, $dr = d\theta = d\phi = 0$, and

$$\Delta\tau = \int\sqrt{-ds^2} \tag{3}$$

$$= \int\left(1-\frac{2GM}{r}\right)^{1/2}dt \tag{4}$$

$$= \left(1-\frac{2GM}{r}\right)^{1/2}\Delta t \tag{5}$$

The proper time interval for an object at rest is thus less than Δt unless we are at an infinite distance from the mass. Note also that proper time appears to stop (in the sense that $\Delta\tau = 0$) when $r = 2GM$, which is the Schwarzschild radius.

The difference between t and τ is difficult to visualize, and I'm not certain I really understand it. In his book, Moore describes an experimental setup in which a clock measuring τ and a 't-meter' measuring t are placed at a fixed point (fixed value of r). The clock measures proper time and we can calculate the reading on the t-meter from it by using the above equation. However, we can also measure t by placing another clock at infinity, where Δt and $\Delta\tau$ are equal. If we send a light signal once a second (as measured at infinity) from this clock towards the t-meter at r, then (ignoring the fact that if the distant clock is at infinity, it will take an infinite time to reach the t-meter) every time the t-meter receives a signal, it advances by 1 second. Since the light signal is travelling through curved space-time, its path isn't the same as a light signal travelling through flat space-time. That is, if we removed the mass, thus making space-time flat, then the clock at r and the t-meter at r would agree. With the mass present, however, the time required by the light to reach the t-meter is increased since the light is following a curved path, so that the t-meter always says that a larger amount of time Δt has elapsed than the interval $\Delta\tau$ measured by the proper time clock attached to the object.

Another way of looking at it is that if we release two light pulses separated by interval Δt_A at $r = r_A$ and detect them at some other point where $r = r_B$, the interval Δt_B measured by the detector will be the same as that at the source: $\Delta t_B = \Delta t_A$, although the *proper* time intervals that elapse at the source and detector will *not* be equal: $\Delta\tau_B \neq \Delta\tau_A$. It's mind-bending stuff and I'm not sure how much benefit can be obtained by trying to visualize what's going on intuitively; probably not much.

Here's another example of a metric (fictitious as far as I know):

$$ds^2 = -dt^2 + dr^2 + R^2 \sinh^2 \frac{r}{R} \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{6}$$

For an object at rest here, the proper time and t coordinate always agree, since

$$ds^2 = -d\tau^2 = -dt^2 \tag{7}$$

This metric describes a spherically symmetric space-time, since if we fix $r = r_0$ (and $t = t_0$) then $dr = dt = 0$ and

$$ds^2 = R^2 \sinh^2 \frac{r_0}{R} \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{8}$$

That is, the metric has the form

$$ds^2 = K^2 d\theta^2 + K^2 \sin^2\theta d\phi^2 \tag{9}$$

for a constant K, which is the same as that of spherical coordinates for constant radius.

The meaning of the r coordinate can be found by choosing a constant $r = r_0$ and $t = t_0$ at a fixed value of $\theta = \frac{\pi}{2}$. Then

$$ds = R \sinh \frac{r_0}{R} d\phi \tag{10}$$

If we now integrate this through the range $0 \leq \phi \leq 2\pi$ we get the circumference C of the equatorial circle:

$$C = 2\pi R \sinh \frac{r_0}{R} \tag{11}$$

The Taylor expansion of $\sinh x$ is

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \tag{12}$$

so for $x > 0$, $\sinh x > x$. Therefore

$$C > 2\pi R \frac{r_0}{R} = 2\pi r_0 \tag{13}$$

Since a diagonal metric tensor means that the basis vectors are orthogonal, the fact that this metric is diagonal means the coordinate system is orthogonal.

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