SCHWARZSCHILD METRIC EQUIVALENT TO WEAK FIELD SOLUTION FOR SPHERICAL OBJECT

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In the weak field, low velocity limit, the perturbation to the flat space metric can be found in terms of the stress-energy tensor

$$h_{jm}=2G\int\frac{2T_{jm}-\eta_{jm}T}{|\mathbf{r}-\mathbf{r}'|}d^{3}\mathbf{r}' \tag{1}$$

For a spherically symmetric object composed of a perfect fluid whose component particles are moving slowly (compared to light), the pressure is negligible and the density is approximately equal to the Newtonian mass density. Using our previous results, we have

$$\rho_{g}=2T_{tt}-\eta_{tt}T=\rho_{0}+3P_{0}\approx\rho_{0} \tag{2}$$

$$\rho_{c}=2T_{ii}-\eta_{ii}T=\rho_{0}-P_{0}\approx\rho_{0} \tag{3}$$

where i is a spatial index in the last line. Off-diagonal h_{jm} components are proportional to the velocity so are all zero if the object is stationary.

Therefore from 1 we get

$$h_{aa}(\mathbf{r})=2G\rho_{0}\int_{\mathcal{V}}\frac{d^{3}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{4}$$

$$=\frac{2G\rho_{0}\mathcal{V}}{r} \tag{5}$$

$$=\frac{2GM}{r} \tag{6}$$

where $\mathcal{V}$ is the volume of the sphere. The metric for such an object is then

$$ds^{2}=g_{tt}dt^{2}+g_{xx}dx^{2}+g_{yy}dy^{2}+g_{zz}dz^{2} \tag{7}$$

$$=(\eta_{tt}+h_{tt})\,dt^{2}+(\eta_{xx}+h_{xx})\,dx^{2}+(\eta_{yy}+h_{yy})\,dy^{2}+(\eta_{zz}+h_{zz})\,dz^{2} \tag{8}$$

$$=-\left(1-\frac{2GM}{r}\right)dt^{2}+\left(1+\frac{2GM}{r}\right)\left(dx^{2}+dy^{2}+dz^{2}\right) \tag{9}$$

In spherical coordinates we have

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{10}$$

so

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{11}$$

The r coordinate is not circumferential, in the sense that the distance around the equator of the sphere is not $2\pi r$. We can see this by noting that to find the equatorial circumference, we hold r constant at $r = R$ and $\theta = \frac{\pi}{2}$, so that $ds = \sqrt{1 + \frac{2GM}{R}} R d\phi$ and

$$C = \int_0^{2\pi} \left(1 + \frac{2GM}{R}\right) R d\phi = 2\pi \sqrt{1 + \frac{2GM}{R}} R \tag{12}$$

We can define an alternative radial coordinate as

$$r_c \equiv \sqrt{1 + \frac{2GM}{r}} r \tag{13}$$

The r_c coordinate is circumferential, since the equatorial circumference is $C = 2\pi R_c$.

Assuming that $\frac{GM}{r}$ is small (equivalent to assuming we're a long way from the object so the field is weak), then

$$dr_c = \sqrt{1 + \frac{2GM}{r}} dr + \left(1 + \frac{2GM}{r}\right)^{-1/2} \left(-\frac{GM}{r^2} dr\right) r \tag{14}$$

$$= \left[\sqrt{1 + \frac{2GM}{r}} - \frac{GM}{r}\left(1 + \frac{2GM}{r}\right)^{-1/2}\right] dr \tag{15}$$

$$\approx \left[1 + \frac{GM}{r} - \frac{GM}{r}\left(1 - \frac{GM}{r}\right)\right] dr \tag{16}$$

$$= dr + \mathcal{O}\left(\left(\frac{GM}{r}\right)^2\right) \tag{17}$$

Thus to first order $dr = dr_c$ so we can write 11 as

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) dr_c^2 + r_c^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{18}$$

The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{r_c}\right)dt^2 + \left(1 - \frac{2GM}{r_c}\right)^{-1} dr_c^2 + r_c^2 d\theta^2 + r_c^2 \sin^2\theta d\phi^2 \tag{19}$$

where we've used the circumferential coordinate r_c for the radial component. We can show that, to first order, 18 and 19 are equivalent. Start with the second term in the Schwarzschild metric:

$$\left(1 - \frac{2GM}{r_c}\right)^{-1} \approx 1 + \frac{2GM}{r_c} \tag{20}$$

$$= 1 + \frac{2GM}{\sqrt{1 + \frac{2GM}{r}}r} \tag{21}$$

$$\approx 1 + \frac{2GM}{r}\left(1 - \frac{GM}{r}\right) \tag{22}$$

$$= 1 + \frac{2GM}{r} + \mathcal{O}\left(\left(\frac{GM}{r}\right)^2\right) \tag{23}$$

The time component also matches, since

$$1 - \frac{2GM}{r_c} = 1 - \frac{2GM}{\sqrt{1 + \frac{2GM}{r}}r} \tag{24}$$

$$\approx 1 - \frac{2GM}{r}\left(1 - \frac{GM}{r}\right) \tag{25}$$

$$= 1 - \frac{2GM}{r} + \mathcal{O}\left(\left(\frac{GM}{r}\right)^2\right) \tag{26}$$

Thus in the weak field limit, the approximate solution is equivalent to the Schwarzschild solution to first order.