SCHWARZSCHILD METRIC WITH NEGATIVE MASS

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In the derivation of the Schwarzschild metric we got to the form

$$ds^2 = -\left(1+\frac{C}{r}\right)dt^2 + \left(1+\frac{C}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (1)$$

where C is a constant of integration, and by requiring the metric to reduce to Newton's gravitation law for large r, we found that $C = -2GM$. However, we can also consider the case where $C > 0$, which is essentially considering the possibility of negative mass.

First, how will a test mass released at rest from a distance r behave? From our previous derivation we have the acceleration given by

$$\frac{d^2r}{d\tau^2} = \frac{C}{2r^2} \qquad (2)$$

Since $C > 0$, the quantity on the RHS is positive, so the particle will accelerate *away* from the origin. Thus, as we might expect, a negative mass repels a positive mass.

Since the object is repelled, we might expect that it could not exist in a closed orbit about the negative mass. For the case of circular orbits, we can see this is true by scanning through the derivation we did earlier for the ordinary Schwarzschild metric, but replacing $-2GM$ by C. In the original metric, the radius of a circular orbit was given by the solutions of the quadratic equation

$$GMr^2 - l^2 r + 3GMl^2 = 0 \qquad (3)$$

where l is the constant angular momentum. Replacing $-2GM$ by C and multiplying through by -1 gives us the quadratic

$$\frac{C}{2}r^2 + l^2 r + \frac{3C}{2}l^2 = 0 \qquad (4)$$

Since all 3 terms in this equation are intrinsically positive, it has no real, positive roots for r, so there are no circular orbits possible.

The ordinary Schwarzschild metric has an event horizon at $r = 2GM$, that is, a point at which $g_{tt} = 0$ and if r crosses this point, the signs of both

g_{tt} and g_{rr} change, causing a swap between the time and radial coordinates. In order for an event horizon to exist if $C>0$, there must be a value of r where $g_{tt}=0$, but since $g_{tt}=-\left(1+\frac{C}{r}\right)$, this does not happen.

The singularity at $r=0$ is still there, however, and is a geometric singularity (that is, one that is a result of the intrinsic geometry of the metric) rather than a coordinate singularity (one that is an artifact of the coordinate system). Because there is no event horizon, this is a *naked singularity*, which can be approached directly without time and space swapping round.

There is one aspect of this metric that still puzzles me however. If we follow through the derivation of the constants of the motion that we did earlier for the Schwarzschild metric but replacing $-2GM$ by C, we get

$$\frac{dt}{d\tau}=e\left(1+\frac{C}{r}\right)^{-1} \tag{5}$$

$$\frac{d\theta}{d\tau}=0 \tag{6}$$

$$\frac{d\phi}{d\tau}=0 \tag{7}$$

$$\frac{dr}{d\tau}=\pm\sqrt{e^2-\left(1+\frac{C}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)} \tag{8}$$

where e and l are constants of the motion.

For the Schwarzschild metric, in general $dt/d\tau=u^t$, the time component of the four-velocity. In this case, t is the time as measured by an observer at rest at infinity, and τ is the proper time as measured by the object, which may be moving. The four-momentum's time component is the energy, and the four-velocity is the four-momentum per unit mass, so e is the energy per unit mass of the object which remains constant as the object moves in from infinity.

If we use the same interpretation for the $C>0$ case and consider a particle released from rest at a distance r, then $\ell=0$, $e=1$ and we end up with $\frac{dr}{d\tau}=\pm\sqrt{-C/r}$ which makes $\frac{dr}{d\tau}$ imaginary. However, if we take the derivative of this, we get

$$\frac{d^2r}{d\tau^2}=-\frac{\sqrt{-C}}{r^{3/2}}\frac{dr}{d\tau}=-\frac{\sqrt{-C}}{r^{3/2}}\frac{\sqrt{-C}}{r^{1/2}}=\frac{C}{2r^2} \tag{9}$$

which agrees with 2.

Something odd is happening with the time coordinate t in this case, since at $r\to\infty$, $\frac{dt}{d\tau}=e$ and as r decreases, so does $\frac{dt}{d\tau}$, reaching zero at $r=0$. This is the opposite behaviour to the t coordinate in the ordinary Schwarzschild metric, where the proper time slows down as we get closer to the event horizon, eventually stopping when $r = 2GM$. Because the time coordinate behaves differently in this case, is it still correct to identify it with the proper time of an observer at infinity? Comments welcome.