SCHWARZSCHILD RADIUS

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Starting from the geodesic equation we derived earlier, we can transform it into a different form which is sometimes more useful. The starting point is

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right)-\frac{1}{2}\partial_a g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \tag{1}$$

We can expand the first term to get

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right)=\frac{dg_{aj}}{d\tau}\frac{dx^j}{d\tau}+g_{aj}\frac{d^2x^j}{d\tau^2} \tag{2}$$

$$=\partial_k g_{aj}\frac{\partial x^k}{\partial\tau}\frac{dx^j}{d\tau}+g_{aj}\frac{d^2x^j}{d\tau^2} \tag{3}$$

We can substitute this back into 1 to get

$$\partial_k g_{aj}\frac{\partial x^k}{\partial\tau}\frac{dx^j}{d\tau}+g_{aj}\frac{d^2x^j}{d\tau^2}-\frac{1}{2}\partial_a g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \tag{4}$$

We can now multiply this by g^{al} and sum over a, using $g^{al}g_{aj}=g^{la}g_{aj}=\delta^l{}_j$

$$\frac{d^2x^l}{d\tau^2}+g^{al}\left[\partial_k g_{aj}\frac{\partial x^k}{\partial\tau}\frac{dx^j}{d\tau}-\frac{1}{2}\partial_a g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]=0 \tag{5}$$

$$\frac{d^2x^l}{d\tau^2}+g^{al}\left(\partial_i g_{aj}-\frac{1}{2}\partial_a g_{ij}\right)\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \tag{6}$$

$$\frac{d^2x^l}{d\tau^2}+g^{al}\left(\partial_i g_{aj}-\frac{1}{2}\partial_a g_{ij}\right)u^iu^j=0 \tag{7}$$

where we've relabelled the dummy index k to i in the second line. Note that the only non-dummy index is l so this is actually a set of n equations, where n is the dimension of the space-time (or just space).

This form of the geodesic equation requires knowing both the contravariant and covariant forms of the metric tensor, but for diagonal metrics $g^{ij}=$

$1/g_{ij}$. For non-diagonal metrics, we do have to calculate a matrix inverse, but this isn't usually too hard.

In the Schwarzschild metric, we can use this form of the geodesic equation to get an equation of motion for the radial coordinate. This metric is

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (8)$$

Since this metric is diagonal, the inverse metric is easy to calculate.

Now suppose that the object is at rest, so that the spatial components of the four velocity $u^i = 0$, for $i = r,\theta,\phi$. From the condition $u^i u_i = -1$ we therefore get

$$u^i u_i = g_{ij} u^i u^j \quad (9)$$

$$-1 = g_{tt}\left(u^t\right)^2 \quad (10)$$

$$= -\left(1-\frac{r_s}{r}\right)\left(u^t\right)^2 \quad (11)$$

$$u^t = \left(1-\frac{r_s}{r}\right)^{-1/2} \quad (12)$$

For the case of an object at rest, the only non-zero terms in the geodesic equation above are those with $i = j = t$, so

$$\frac{d^2x^l}{d\tau^2} + g^{al}\left(\partial_t g_{at} - \frac{1}{2}\partial_a g_{tt}\right)\left(1-\frac{r_s}{r}\right)^{-1} = 0 \quad (13)$$

Since none of the metric components depends on t, the term $\partial_t g_{at}$ is zero. The term $\frac{1}{2}\partial_a g_{tt}$ is non-zero only if $a = r$, in which case we have

$$\frac{1}{2}\partial_r g_{tt} = -\frac{r_s}{2r^2} \quad (14)$$

The equation of motion then becomes

$$\frac{d^2x^l}{d\tau^2} + g^{rl}\frac{r_s}{2r^2}\left(1-\frac{r_s}{r}\right)^{-1} = 0 \quad (15)$$

The second term is non-zero only when $l = r$, and $g^{rr} = 1/g_{rr}$ so we get

$$\frac{d^2x^r}{d\tau^2} + \left(1-\frac{r_s}{r}\right)\frac{r_s}{2r^2}\left(1-\frac{r_s}{r}\right)^{-1} = 0 \quad (16)$$

$$\frac{d^2r}{d\tau^2} = -\frac{r_s}{2r^2} \quad (17)$$

If we require this to reduce to the Newtonian theory of gravity for large distances, where $r \gg r_s$ then when an object starts at rest a distance r from an object of mass M, its initial acceleration is

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2} \tag{18}$$

In the Newtonian limit, $\tau \to t$ and $r \gg r_s$ so if the correspondence is to work, we must have

Schwarzschild radius

$$\boxed{r_s = 2GM} \tag{19}$$

This value is known as the *Schwarzschild radius*.

Example 1. Suppose we have a neutron star with mass M. If we consider a shell around the star with a circumference of $C_1 = 6\pi GM$, then the radial coordinate is $r_1 = C_1/2\pi = 3GM$. There is another shell with circumference $C_2 = 20\pi GM$, and radial coordinate $r_2 = 10GM$. The radial distance between these shells is then

$$\Delta r = \left[\sqrt{r(r-r_s)} + \frac{r_s}{2}\ln\left[\sqrt{r(r-r_s)} + r - \frac{r_s}{2}\right]\right]_{r_1}^{r_2} \tag{20}$$

$$= \sqrt{80}GM + GM\ln\left(\sqrt{80}GM + 9GM\right) - \left(\sqrt{3}GM + GM\ln\left(\sqrt{3}GM + 2GM\right)\right) \tag{21}$$

$$= GM\left(\sqrt{80} - \sqrt{3} + \ln\left(\frac{\sqrt{80}+9}{\sqrt{3}+2}\right)\right) \tag{22}$$

$$= 8.7825GM \tag{23}$$

Note this is greater than the ordinary difference in flat space of $r_2 - r_1 = 7GM$.

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