SPACETIME DIAGRAMS - EVENTS AND WORLD LINES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 27 June 2021.

Here's a detailed look at how various events and world lines look in a spacetime diagram. We look at the diagrams for two observers. The first observer $\mathcal{O}$ is stationary in the frame of the diagrams, and a second observer $\bar{\mathcal{O}}$ moves with velocity $v = 0.5$ in the $+x$ direction. As usual, the coordinate systems of the two observers agree at $t = \bar{t} = x = \bar{x} = 0$. First, we consider a clock in $\mathcal{O}$'s system at rest at $x = 1$ (we'll assume all distances and times are measured in metres). Since the clock is at rest, its x coordinate remains constant, so we have Fig. 1.

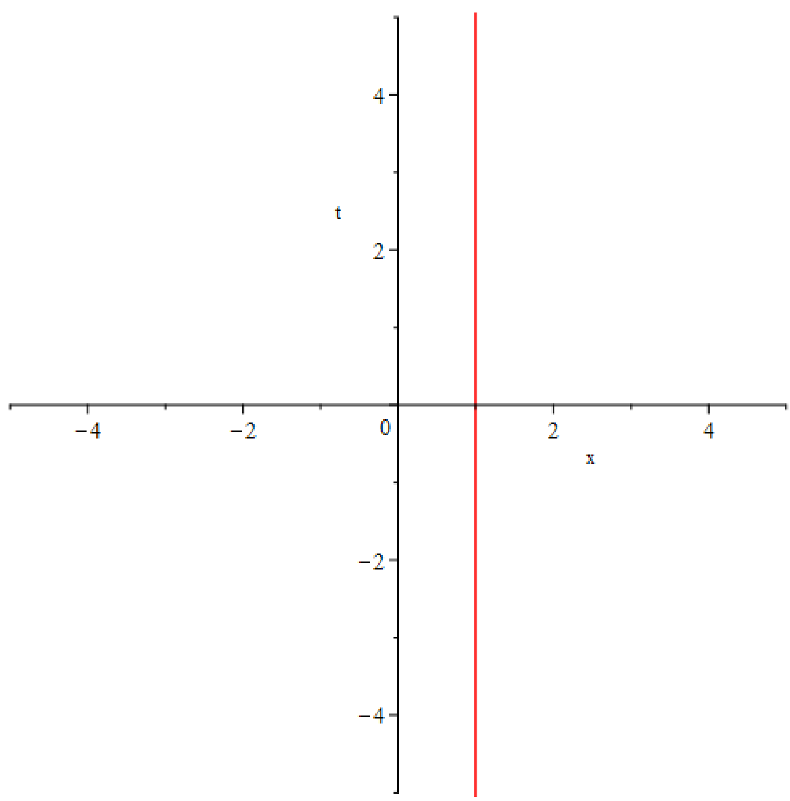

FIGURE 1. World line of clock at $x = 1$.

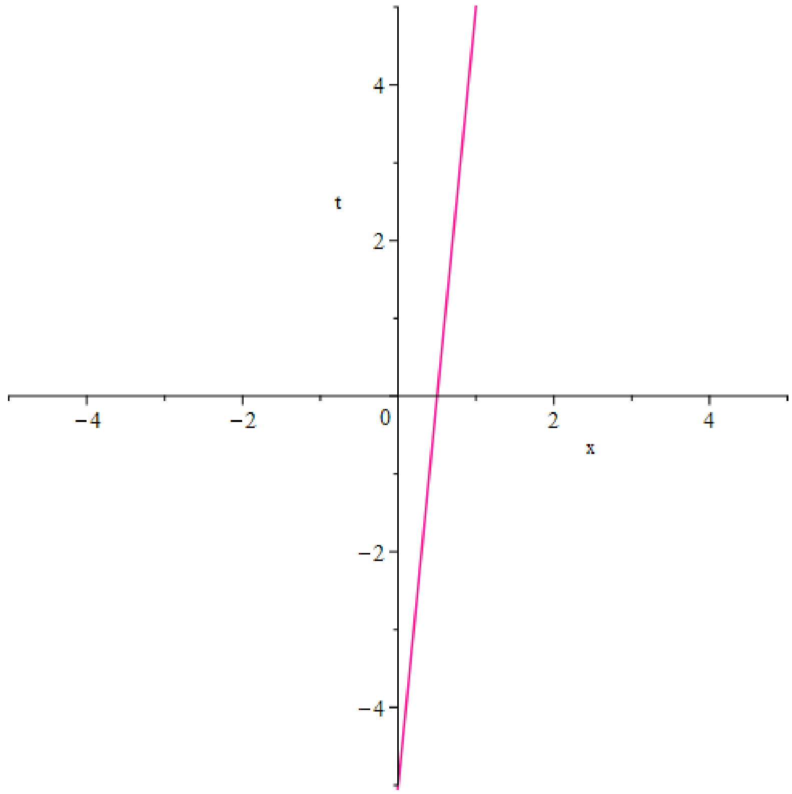

FIGURE 2. World line of object with $v = 0.5$, passing through the event $(x,t) = (0.5,0)$.

Now consider an object with velocity $\frac{dx}{dt} = 0.5$ which is at $x = 0.5$ at $t = 0$. We are still in $\mathcal{O}$'s frame, so the world line of this object is shown in Fig. 2.

The $\bar{t}$ and $\bar{x}$ axes of observer $\bar{\mathcal{O}}$ will have slopes of $\frac{1}{0.5} = 2$ and 0.5 respectively, as we found earlier. This is shown in Fig. 3.

Returning to $\mathcal{O}$, we can now find the locus of points on the diagram corresponding to events with a spacetime distance from the origin of $\Delta s^2 = -1$. Since we're considering distances from the origin, this condition is equivalent to the equation

$$x^2 - t^2 = -1 \tag{1}$$

which is the equation of a hyperbola, as shown in Fig. 4.

Similarly, we can plot points with $\Delta s^2 = +1$. This is equivalent to

$$x^2 - t^2 = 1 \tag{2}$$

and gives the hyperbolas shown in Fig. 5.

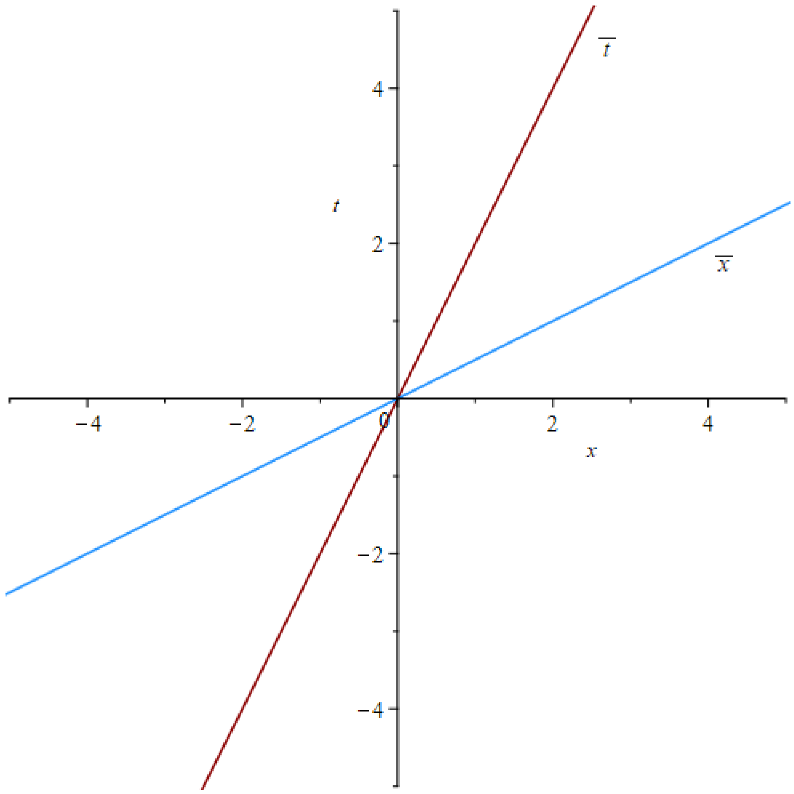

FIGURE 3. Axes of observer $\bar{\mathcal{O}}$ as seen by $\mathcal{O}$.

Returning to observer $\bar{\mathcal{O}}$, we need to find the points along the $\bar{t}$ and $\bar{x}$ axes that correspond to 1m intervals, as measured by $\bar{\mathcal{O}}$. The key to finding these is that the intervals $\Delta s^2 = \pm n^2$ where n is an integer number of metres, are invariant, so they are the same to both observers. Thus if we draw the hyperbolas corresponding to these intervals, the calibration points on the $\bar{t}$ and $\bar{x}$ axes are the points where these axes intersect the hyperbolas. This is shown in Fig. 6.

Now consider the locus of points with separation $\Delta s^2 = 0$ from the origin. These are the world lines of light rays passing through the origin, so they are given by the straight lines $t = \pm x$, as shown in Fig. 7.

The locus of all events that occur at $t = 2$ comprises those events that are simultaenous as seen by $\mathcal{O}$. Thus they will lie on a horizontal line, as in Fig. 8.

Now we consider events that occur at $\bar{t} = 2$, so are simultaneous as seen by $\bar{\mathcal{O}}$. These will lie on the straight line parallel to the $\bar{x}$ axis and intersecting the $\bar{t}$ axis at the point where the hyperbola $x^2 - t^2 = -2^2$ (see Fig. 6) intersects the $\bar{t}$ axis. Thus we have Fig. 9.

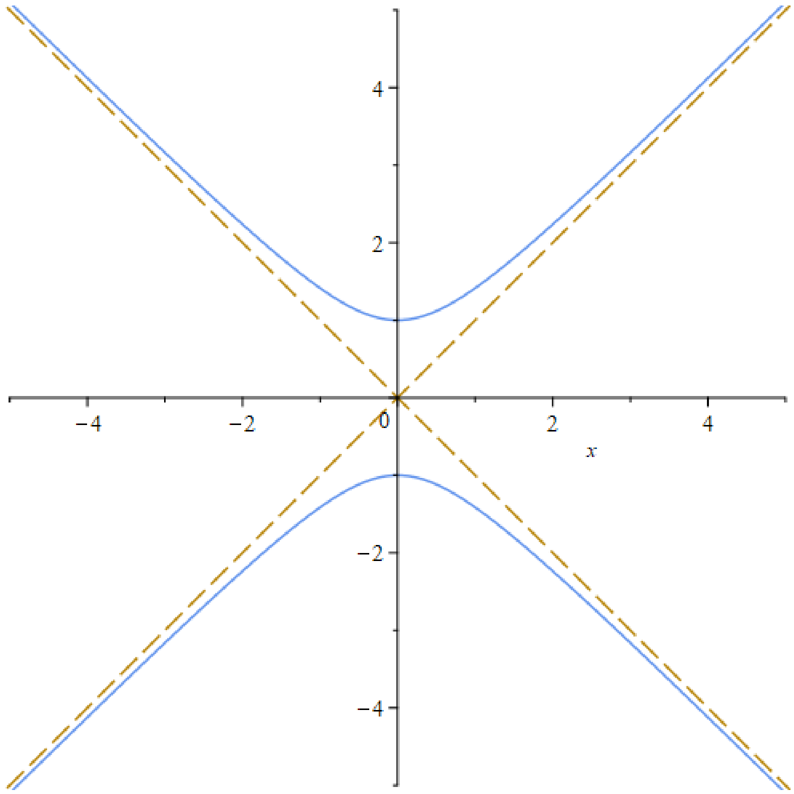

FIGURE 4. The blue hyperbolas represent $\Delta s^2 = -1$. The dashed lines are the asymptotes, and have equations $t = \pm x$.

The specific event that occurs at $\bar{t} = 0$ and $\bar{x} = 0.5$ lies on the hyperbola $x^2 - t^2 = 0.5^2 = 0.25$ and is at the intersection of this hyperbola with the $\bar{x}$ axis, so we have Fig. 10.

The locus of events with $\bar{x} = 1$ is a straight line parallel to the $\bar{t}$ axis passing through the intersection of the hyperbola $x^2 - t^2 = 1$ and the $\bar{x}$ axis, as in Fig. 11.

Finally, we show the path of a photon which is emitted at $(t, x) = (-1, 0)$ (in the $\mathcal{O}$ frame), travels in the $-x$ direction, is reflected by a mirror stationary in the $\bar{\mathcal{O}}$ frame at $\bar{x} = -1$, and is finally absorbed by a detector stationary in the $\mathcal{O}$ frame at $x = 0.75$. The key point here is that a photon's world line is always parallel to one of the lines $t = \pm x$. The situation is as shown in Fig. 12.

We could just sketch in the paths, but it's instructive to calculate the actual numerical values of the various events. The event of the emission of the photon is at $(t, x) = (-1, 0)$, which is at $t = -1$ on the t axis, as shown by the lower end of the thick yellow line. The photon travels in the $-x$

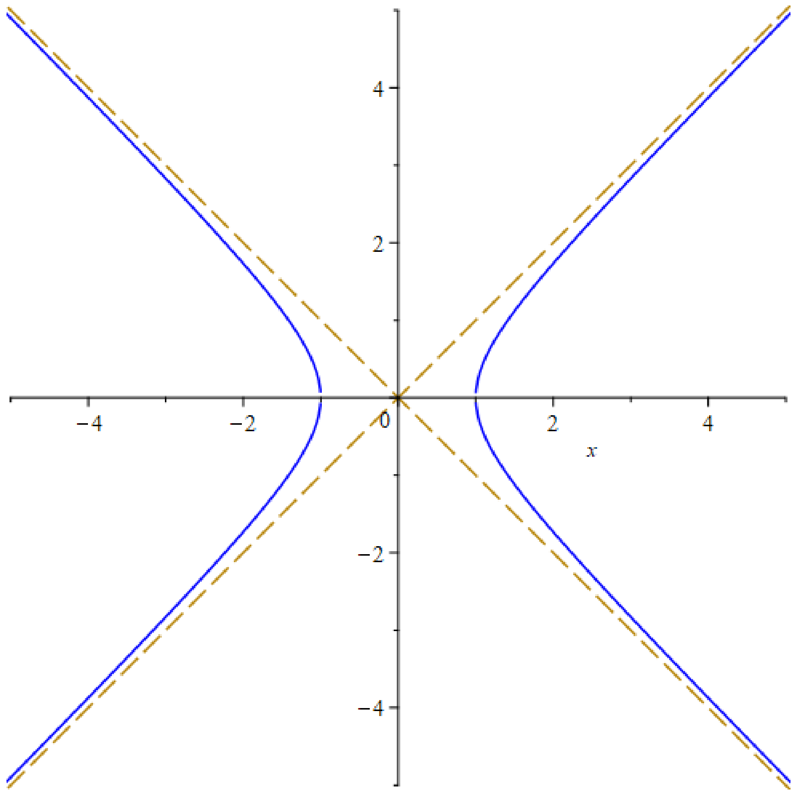

FIGURE 5. The blue hyperbolas represent $\Delta s^2 = +1$. The dashed lines are the asymptotes, and have equations $t = \pm x$.

direction, so its world line is

$$t = -x - 1 \tag{3}$$

The reflection occurs at $\bar{x} = -1$, which is the red line. To get the equation of this line, we can find the point of intersection of the blue hyperbola ($x^2 - t^2 = 1$) with the $\bar{x}$ axis ($t = 0.5x$). Since I used Maple to generate the plots, I also used it to solve the equations. The intersection occurs at $(t, x) = (-0.577, -1.155)$. The red line thus passes through this point and has slope 2, so its equation is (using the point-slope equation for a straight line $t - t_0 = m(x - x_0)$ where m is the slope and (t_0, x_0) is the point on the line):

$$t - (-0.577) = 2(x - (-1.155)) \tag{4}$$

$$t = 1.732 + 2x \tag{5}$$

The event of the photon being reflected is then the intersection of the red line 5 and the lower yellow line 3. This point works out to

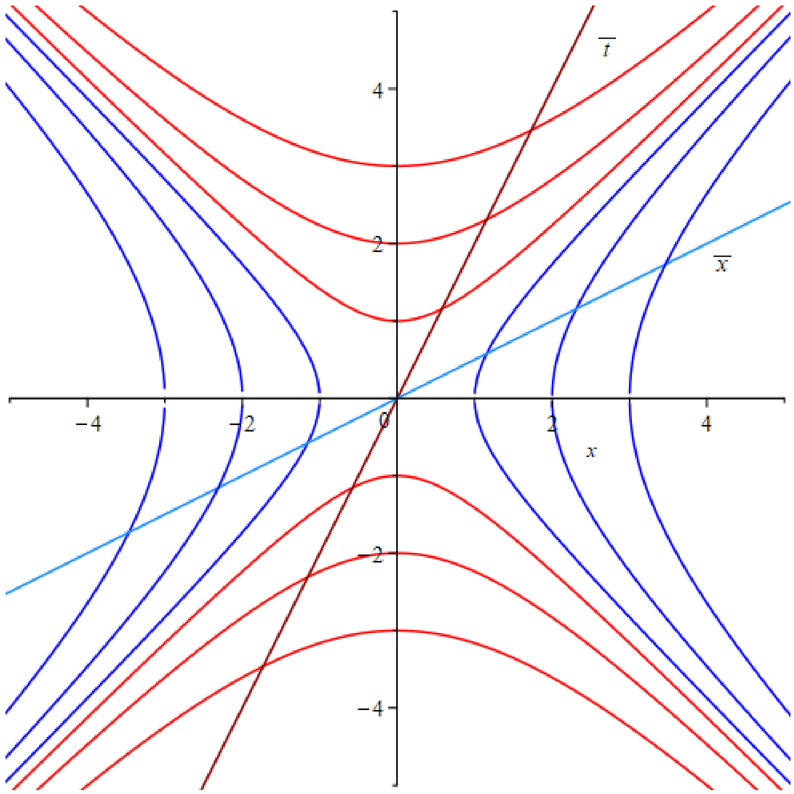

FIGURE 6. The red hyperbolas are $x^2 - t^2 = -n^2$ and the blue hyperbolas are $x^2 - t^2 = n^2$. Their intersections with the $\bar{t}$ axis (brown) and $\bar{x}$ axis (light blue) are the calibration marks at 1m intervals on the two axes.

$$(t,x) = (-0.089, -0.911) \tag{6}$$

The path of the reflected photon is thus the straight line with slope +1 that passes through this point (top yellow line in Fig. 12). This line has the equation

$$t - (-0.089) = x - (-0.911) \tag{7}$$

$$t = x + 0.821 \tag{8}$$

The event of absorption by the detector (top end of upper yellow line) occurs at the intersection of this line with the vertical line $x = 0.75$ and thus occurs at

$$(t,x) = (1.571, 0.75) \tag{9}$$

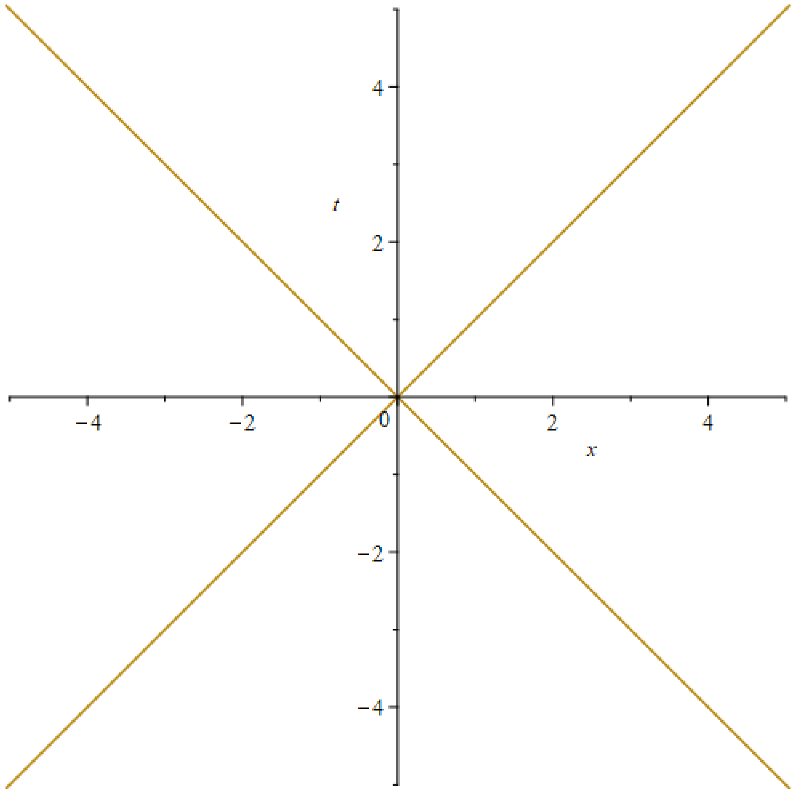

FIGURE 7. World lines (brown) of light rays passing through the origin.

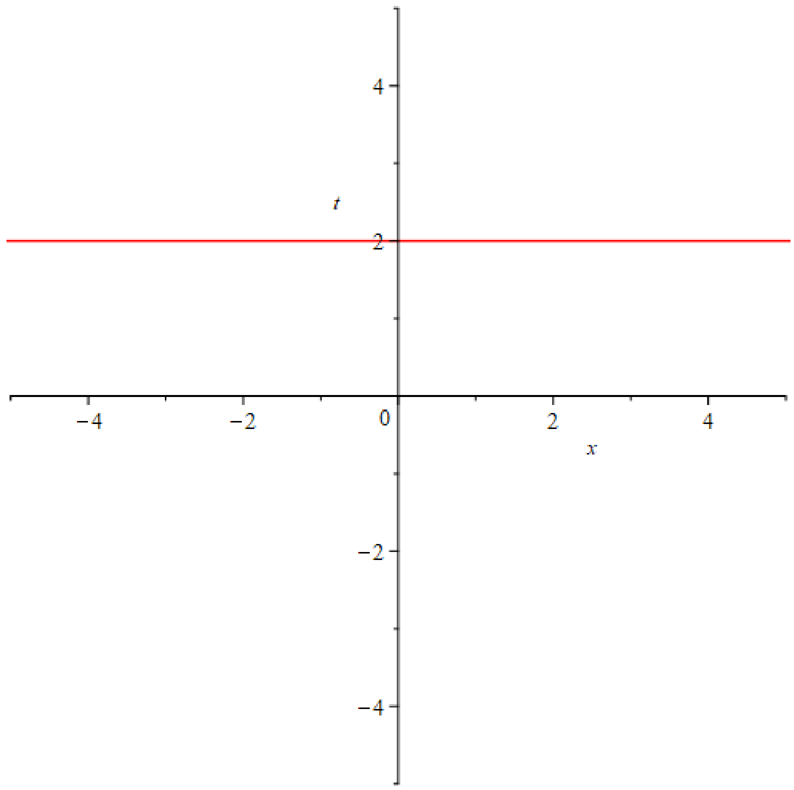

FIGURE 8. Events that are simultaneous at $t = 2$, as seen by $\mathcal{O}$.

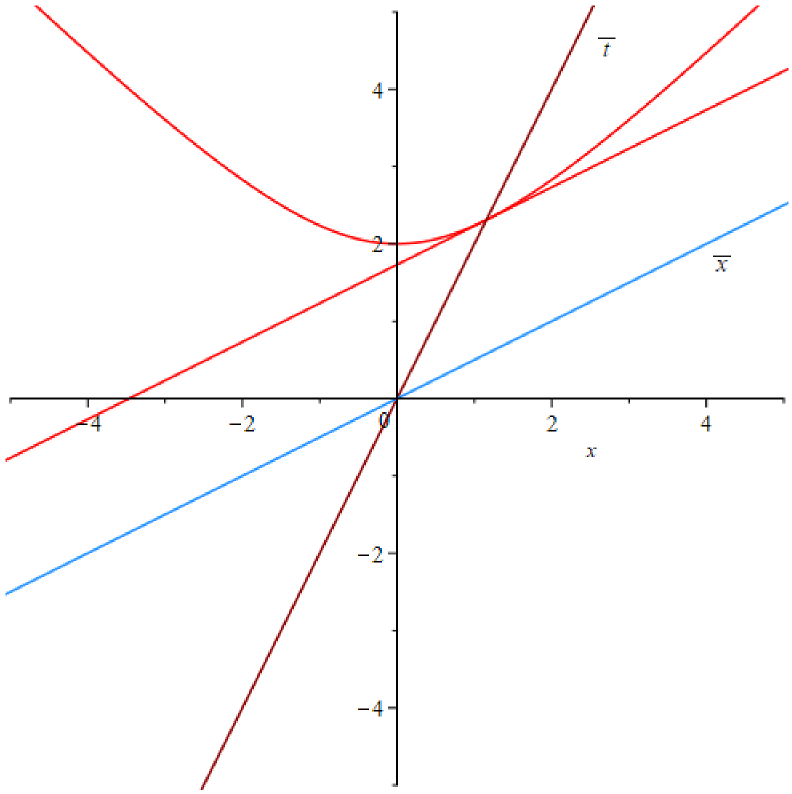

FIGURE 9. Locus of points $\bar{t} = 2$ (straight red line) intersects the $\bar{t}$ axis at the same point as the hyperbola $x^2 - t^2 = -2^2$.

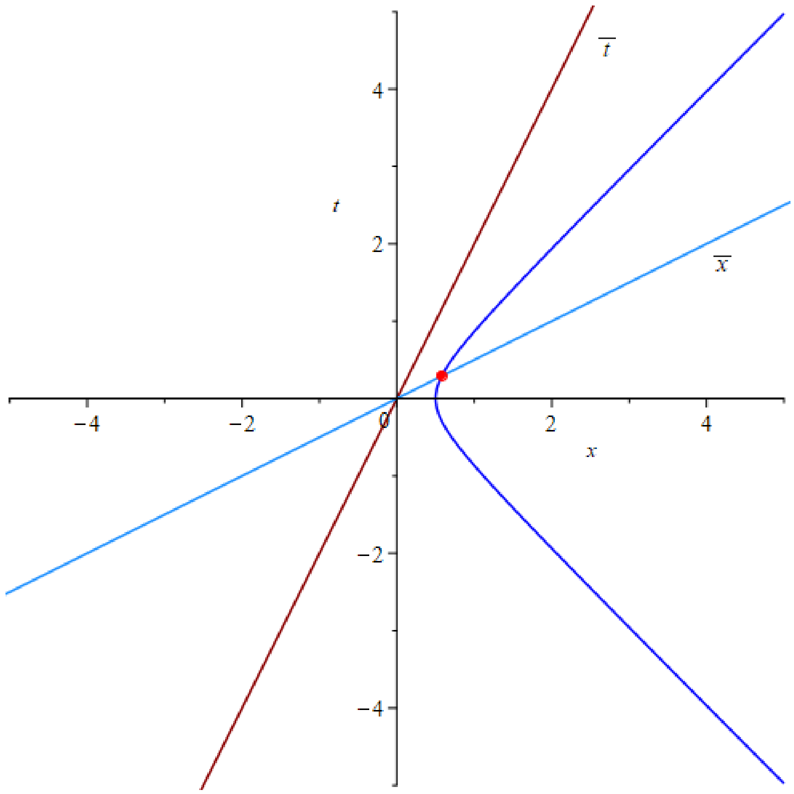

FIGURE 10. The blue hyperbola is the curve $x^2 - t^2 = 0.5^2$. Its intersection (red dot) with the $\bar{x}$ axis is the event $(\bar{t}, \bar{x}) = (0, 0.5)$.

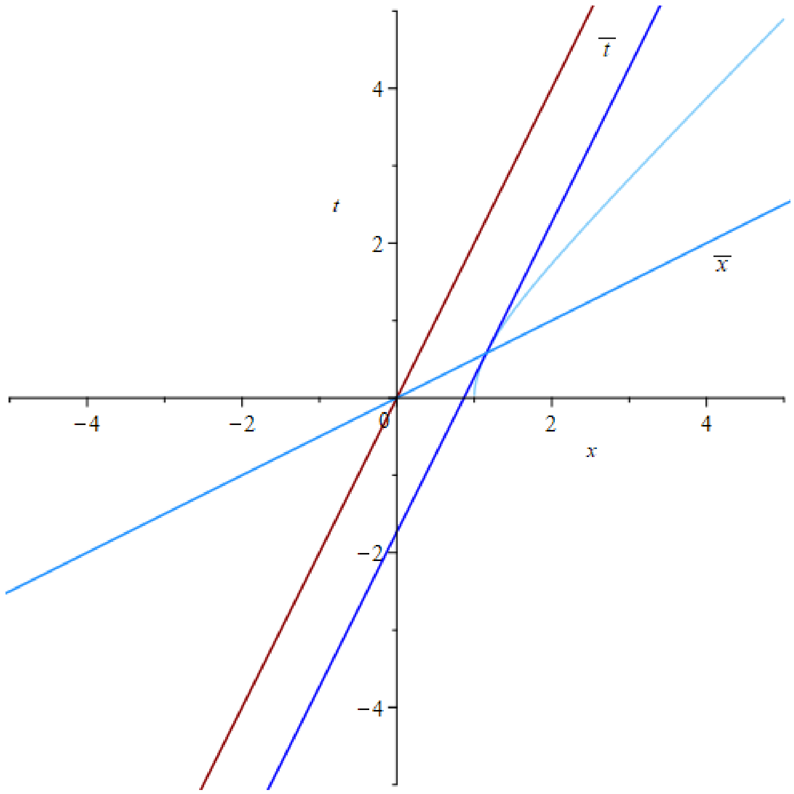

FIGURE 11. Locus of events with $\bar{x} = 1$ (dark blue line parallel to $\bar{t}$ axis). The light blue hyperbola is the curve $x^2 - t^2 = 1$.

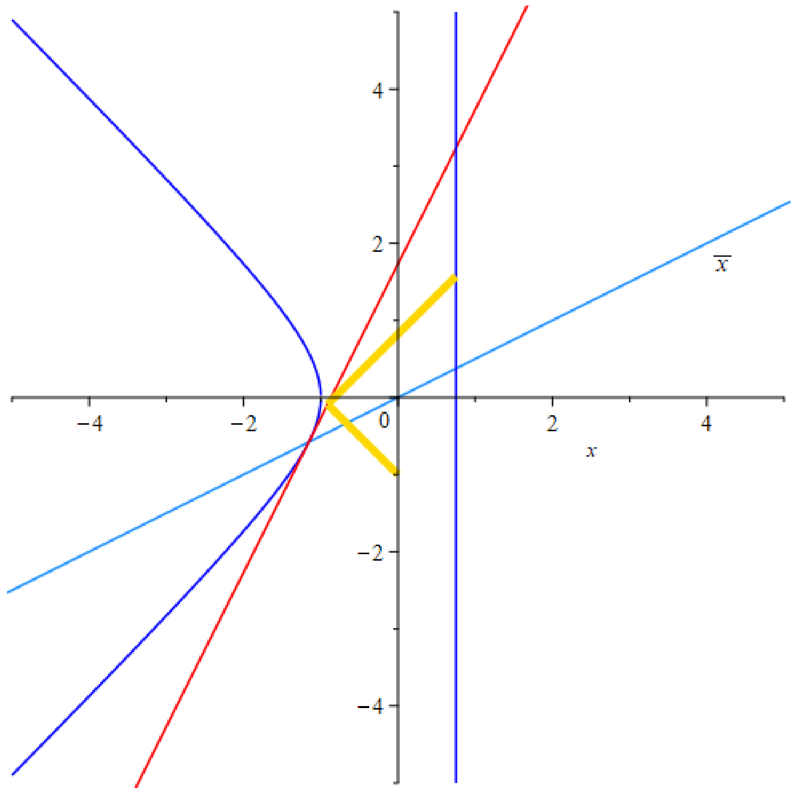

FIGURE 12. Path of a photon that is reflected and then absorbed.