SQUASHED SPHERE

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We consider a 'squashed sphere' with the metric

$$ds^2 = \left(b^2 + a^2\cos^2\theta\right)d\theta^2 + \frac{\left(b^2+a^2\right)^2}{b^2+a^2\cos^2\theta}\sin^2\theta\, d\phi^2 \tag{1}$$

where a and b are constants, and the angle variables have the usual ranges: $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

We can find the length of the equator and of a circle of fixed longitude going through the two poles. For the equator $\theta = \frac{\pi}{2}$ and $d\theta = 0$, so the length element is

$$ds = \frac{b^2+a^2}{b}d\phi \tag{2}$$

so the circumference C_E of the equator is

$$C_E = \int_0^{2\pi}\frac{b^2+a^2}{b}d\phi = \frac{2\pi}{b}\left(b^2+a^2\right) \tag{3}$$

For a polar circle, we can take $\phi = 0$, $d\phi = 0$ and let θ vary from 0 to π, which gives us a semicircle, so we double this result to get the full circle. The polar circumference is then C_P

$$C_P = 2\int_0^{\pi} d\theta\sqrt{b^2+a^2\cos^2\theta} \tag{4}$$

Running this integral through Maple, we find that it is a complete elliptic integral, denoted by EllipticE with a single argument. If we set $b = a$, then we get

$$C_P = 2a \times 2\sqrt{2}\text{EllipticE}\left(\frac{\sqrt{2}}{2}\right) \tag{5}$$

This has a numerical value of approximately

$$C_P \approx 7.64a \tag{6}$$

To calculate the area, we use the formula

$$A=\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sqrt{g} \tag{7}$$

where the metric tensor can be read off from 1

$$g_{\mu\nu}=\begin{bmatrix} b^2+a^2\cos^2\theta & 0 \\ 0 & \frac{\left(b^2+a^2\right)^2}{b^2+a^2\cos^2\theta}\sin^2\theta \end{bmatrix} \tag{8}$$

The determinant is

$$g=\left(b^2+a^2\right)^2\sin^2\theta \tag{9}$$

so we have for the area

$$A=\left(b^2+a^2\right)\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta \tag{10}$$

$$=4\pi\left(b^2+a^2\right) \tag{11}$$