STEREOGRAPHIC PROJECTION OF THE SPHERE

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We consider the stereographic projection of a sphere of radius L onto a plane that is tangent to the sphere's south pole (as in Zee's figure 3 in chapter I.5 of *Einstein Gravity*). The projection is obtained by drawing a line from the north pole through each point on the sphere and projecting it until it intersects the plane tangent to the south pole. In this projection, the north pole gets mapped to infinity.

We can rewrite the metric for the original sphere in the form

$$ds^2 = \frac{dr^2}{1-r^2/L^2} + r^2 d\phi^2 \tag{1}$$

We can see that this is the same as the usual metric with

$$r \equiv L\sin\theta \tag{2}$$

so that r represents the radius of a horizontal cross section of the sphere (and not the sphere's radius). In this case

$$dr^2 = L^2\cos^2\theta d\theta^2$$

so 1 becomes

$$ds^2 = \frac{L^2\cos^2\theta d\theta^2}{1-\sin^2\theta} + L^2\sin^2\theta d\phi^2 \tag{3}$$

$$= L^2 d\theta^2 + L^2\sin^2\theta d\phi^2 \tag{4}$$

The projection maps a point (r,ϕ) on the sphere onto a point (ρ,ϕ) on the plane. Since each line of projection is parallel to a line of longitude on the sphere, the angle ϕ is unchanged by the projection.

By referring to Zee's figure, we can compare the two triangles with the upper vertex at the north pole. We see that the upper triangle with r as its horizontal side is similar (in the geometric sense) to the triangle with ρ as its horizontal side. The vertical side of the upper triangle is $L - L\cos\theta$ and the vertical side of the lower triangle is $2L$. Therefore

$$\frac{r}{\rho} = \frac{L-L\cos\theta}{2L} = \frac{1-\cos\theta}{2} \tag{5}$$

Also from the diagram, we see that

$$\sin\theta = \frac{r}{L} \tag{6}$$

so

$$\cos\theta = \sqrt{1 - \frac{r^2}{L^2}} \tag{7}$$

Substituting into 5 and expanding, we have

$$\left(1 - \frac{2r}{\rho}\right)^2 = 1 - \frac{r^2}{L^2} \tag{8}$$

$$\left(\frac{1}{L^2} + \frac{4}{\rho^2}\right) r^2 - \frac{4r}{\rho} = 0 \tag{9}$$

Ignoring the $r = 0$ solution, we get

$$r = \frac{4}{\rho}\left(\frac{\rho^2 L^2}{\rho^2 + 4L^2}\right) \tag{10}$$

$$= \frac{\rho}{1 + \rho^2/4L^2} \tag{11}$$

To get the metric, we need to work out dr in terms of $d\rho$. We have

$$dr = \left[\frac{1}{1 + \frac{\rho^2}{4L^2}} - \frac{\rho^2}{2L^2\left(1 + \frac{\rho^2}{4L^2}\right)^2}\right] d\rho \tag{12}$$

Substituting this and 11 into 1 we get a rather messy expression, which I used Maple to simplify. We have

$$\frac{dr^2}{1 - r^2/L^2} = \left(\left(1 + \frac{\rho^2}{4L^2}\right)^{-1} - \frac{\rho^2}{2L^2}\left(1 + \frac{\rho^2}{4L^2}\right)^{-2}\right)^2 \div$$

$$\left(1 - \frac{\rho^2}{L^2}\left(1 + \frac{\rho^2}{4L^2}\right)^{-2}\right)^{-1} d\rho^2 \tag{13}$$

$$= \frac{16L^4}{(4L^2 + \rho^2)^2} d\rho^2 \tag{14}$$

$$= \frac{1}{\left(1 + \frac{\rho^2}{4L^2}\right)^2} d\rho^2$$

The metric of the sphere in terms of the projected coordinates is (again using 11)

$$ds^2 = \frac{1}{\left(1+\frac{\rho^2}{4L^2}\right)^2} d\rho^2 + r^2 d\phi^2 \tag{15}$$

$$= \frac{1}{\left(1+\frac{\rho^2}{4L^2}\right)^2} \left(d\rho^2 + \rho^2 d\phi^2\right) \tag{16}$$

Note that this is the metric of the surface of the original sphere, and not of the projection. The projection is (by definition, really) flat, while the sphere's surface is curved. Note also that the component $d\rho^2 + \rho^2 d\phi^2$ is just the usual metric in polar coordinates of the plane, so it represents a flat space.

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