STRESS-ENERGY TENSOR - CONSERVATION EQUATIONS

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We can express conservation of energy and momentum in terms of the stress-energy tensor. Recall that the physical meaning of the component T^{tt} is the energy density.

To get the conservation laws, consider a small box with dimensions dx, dy and dz, and restrict our attention to the case of 'dust', that is, a fluid containing particles that are all at rest relative to each other. In that case, the tensor has the form

$$T^{ij} = \rho_0 u^i u^j \tag{1}$$

where ρ_0 is the energy density of the dust in its own rest frame and u^i is the four-velocity of the fluid as measured in some observer's local inertial frame. Then the flow of energy into the box over, say, the left-hand face perpendicular to the x axis at position x in time dt is the energy density multiplied by the velocity component in the x direction v^x.

$$dE_x = \left(T_x^{tt} v^x dt\right) dydz \tag{2}$$

We've multiplied by $dydz$ since this is the area of the face of the box through which the energy is flowing, and thus the total flow of energy is the density multiplied by the volume that crosses the box's face, which is $v^x dtdydz$. The subscript x on T_x^{tt} means that the tensor is evaluated at position x. We have

$$T^{tt} = \rho_0 u^t u^t \tag{3}$$

and $u^t v^x = \gamma v^x = u^x$, so $T^{tt} v^x = T^{tx}$ using 1, and thus

$$dE_x = \left(T_x^{tx} dt\right) dydz \tag{4}$$

Similarly, the energy flowing across the face at position $x + dx$ is then

$$dE_{x+dx} = \left(T_{x+dx}^{tx} dt\right) dydz \tag{5}$$

Taking the difference of these two equations we get

$$\left(T_{x+dx}^{tx} - T_x^{tx}\right) dtdydz = \partial_x T^{tx} dxdtdydz \tag{6}$$

We can write similar equations for the y and z directions:

$$\left(T_{y+dy}^{ty}-T_{y}^{ty}\right)dtdxdz = \partial_{y}T^{ty}dxdtdydz \tag{7}$$

$$\left(T_{z+dz}^{tz}-T_{z}^{tz}\right)dtdydx = \partial_{z}T^{tz}dxdtdydz \tag{8}$$

Adding these up gives the net total change in energy within the boxes

$$dE = -\left(\partial_{x}T^{tx}+\partial_{y}T^{ty}+\partial_{z}T^{tz}\right)dxdydzdt \tag{9}$$

The minus sign occurs because if, say, $\partial_{x}T^{tx}<0$, this indicates that $T_{x}^{tx}>T_{x+dx}^{tx}$ so more energy flows in at position x than flows out at position $x+dx$, resulting in $dE>0$.

The net change in energy due to its flow across the boundaries of the box must be reflected in the change of the energy within the box. The energy density is given by T^{tt} so we must have

$$dE = \left(T_{t+dt}^{tt}-T_{t}^{tt}\right)dxdydz \tag{10}$$

$$= \partial_{t}T^{tt}dxdydzdt \tag{11}$$

The energy conservation law is then given by

$$\partial_{t}T^{tt}dxdydzdt = -\left(\partial_{x}T^{tx}+\partial_{y}T^{ty}+\partial_{z}T^{tz}\right)dxdydzdt \tag{12}$$

Since this must be true for any choice of differentials, the energy conservation law is expressed in the compact form

$$\partial_{j}T^{tj}=0 \tag{13}$$

We can do a similar argument for momentum. The component T^{tj} (where j is a spatial coordinate) is the density of the j component of momentum and the components T^{ij} are the rates of flow of the j component of momentum in the i direction, so the net change in momentum component j due to differences in the flow rate at the boundaries of the box is

$$dp^{j}=-\left(\partial_{x}T^{xj}+\partial_{y}T^{yj}+\partial_{z}T^{zj}\right)dxdydzdt \tag{14}$$

This must be equal to the net change of p^{j} within the box over time dt, so

$$dp^{j}=\partial_{t}T^{tj}dxdydzdt \tag{15}$$

and

$$\partial_{i}T^{ij}=0 \tag{16}$$

This is therefore true for all four values of j and represents conservation of energy and momentum, or just four-momentum.

We derived this formula for the special case of a local inertial frame (LIF). We've seen that the appropriate generalization of the gradient is the absolute gradient or covariant derivative, so the appropriate tensor equation for conservation of four-momentum is

$$\boxed{\nabla_i T^{ij} = 0} \tag{17}$$

In terms of Christoffel symbols, this is

$$\nabla_i T^{ij} = \partial_i T^{ij} + \Gamma^i_{ik} T^{kj} + \Gamma^j_{ik} T^{ik} = 0 \tag{18}$$

We can apply this equation to the more general case of a perfect fluid in general coordinates, where the tensor is

$$T^{ij} = (\rho_0 + P_0) u^i u^j + P_0 g^{ij} \tag{19}$$

We can work out the covariant derivative in a LIF. In a LIF, all Christoffel symbols are zero so we get

$$\nabla_i T^{ij} = \partial_i T^{ij} \tag{20}$$

$$0 = u^i u^j \partial_i (\rho_0 + P_0) + (\rho_0 + P_0) \left[u^i \partial_i u^j + u^j \partial_i u^i\right] + \eta^{ij} \partial_i P_0 \tag{21}$$

The four-velocity always satisfies the relation $\mathbf{u} \cdot \mathbf{u} = u_j u^j = -1$ so we have

$$\begin{aligned}
\partial_i (\mathbf{u} \cdot \mathbf{u}) &= \partial_i \left(u^j u_j\right) && (22)\\
&= \partial_i \left(\eta_{jk} u^k u^j\right) && (23)\\
&= \eta_{jk} \left[u^k \partial_i u^j + u^j \partial_i u^k\right] && (24)\\
&= u_j \partial_i u^j + u_k \partial_i u^k && (25)\\
&= 2 u_j \partial_i u^j && (26)\\
&= 0 && (27)
\end{aligned}$$

We can now multiply 21 by u_j and use the above result to get

$$u^i u_j u^j \partial_i (\rho_0 + P_0) + (\rho_0 + P_0) \left[u^i u_j \partial_i u^j + u_j u^j \partial_i u^i \right] + \eta^{ij} u_j \partial_i P_0 = \tag{28}$$

$$-u^i \partial_i (\rho_0 + P_0) - (\rho_0 + P_0) \partial_i u^i + u^i \partial_i P_0 = 0 \tag{29}$$

$$(\rho_0 + P_0) \partial_i u^i + u^i \partial_i \rho_0 = 0 \tag{30}$$

$$\partial_i \left(u^i \rho_0 \right) + P_0 \partial_i u^i = 0 \tag{31}$$

This last equation is known as the *equation of continuity*. Note that it is valid only in a LIF, since the derivative isn't covariant.

Now we can multiply 30 by $u^j$and subtract it from 21:

$$\begin{aligned} u^i u^j \partial_i P_0 + (\rho_0 + P_0) u^i \partial_i u^j + \eta^{ij} \partial_i P_0 &= 0 && (32) \\ (\rho_0 + P_0) u^i \partial_i u^j &= -\left(u^i u^j + \eta^{ij} \right) \partial_i P_0 && (33) \end{aligned}$$

This is the *equation of motion*, also valid in a LIF.

In the non-relativistic limit, the density in any LIF will be the same, as will the pressure. Also, $P_0 \ll \rho_0$ so we can approximate 31 by

$$\partial_i \left(u^i \rho_0 \right) \approx 0 \tag{34}$$

Using $\mathbf{u} \approx [1, v^x, v^y, v^z]$ this becomes

$$\partial_t \rho_0 = -\vec{\nabla} \cdot (\rho_0 \mathbf{v}) \tag{35}$$

where the arrow above $\vec{\nabla}$ indicates this is the 3-d gradient, not the covariant derivative. This is the Newtonian equation of continuity for a perfect fluid, which expresses conservation of mass.

We can also approximate 33 by neglecting any products of velocity components, since $v^i v^j \ll 1$ if both i and j are spatial coordinates. The LHS becomes

$$\begin{aligned} (\rho_0 + P_0) u^i \partial_i u^j &\approx \rho_0 u^i \partial_i u^j && (36) \\ &= \rho_0 \left[\partial_t v^j + \left(\vec{\mathbf{v}} \cdot \vec{\nabla} \right) v^j \right] && (37) \end{aligned}$$

The term with $j = t$ drops out, since $u^t = 1$ and its derivatives are all zero. We can combine the three spatial coordinates into a single vector expression:

$$\rho_0\left[\partial_t\vec{\mathbf{v}}+\left(\vec{\mathbf{v}}\cdot\vec{\nabla}\right)\vec{\mathbf{v}}\right] \tag{38}$$

The RHS is

$$-\left(u^i u^j+\eta^{ij}\right)\partial_i P_0 \tag{39}$$

If $j=t$, the $i=t$ term in the sum is zero because $u^t u^t+\eta^{ij}=+1-1=0$. If we ignore all other terms that are second order or higher in v and/or P_0, we are left with only $-\eta^{ij}\partial_i P_0$. Looking at the 3 terms with j being a spatial coordinate, this is $-\vec{\nabla}P_0$ so we get the approximation

$$\rho_0\left[\partial_t\vec{\mathbf{v}}+\left(\vec{\mathbf{v}}\cdot\vec{\nabla}\right)\vec{\mathbf{v}}\right]=-\vec{\nabla}P_0 \tag{40}$$

which is Euler's equation of motion for a perfect fluid.

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