STRESS-ENERGY TENSOR - SYMMETRY

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Post date: 16 Feb 2021.

We've looked at the stress-energy tensor $T^{\alpha\beta}$ for the special cases of dust and perfect fluids. The components of the stress-energy tensor have the following meanings:

- T^{00} is the energy density, that is the energy per unit volume. 'Energy' is defined in the relativistic sense, so it includes both mass and other forms of energy due to motion and forces.
- T^{0i} is the energy per unit time that crosses a unit area perpendicular to the x^i direction.
- T^{i0} is the momentum in the x^i direction in a unit volume that crosses a unit area of constant time x^0. In other words, T^{i0} is the momentum density.
- T^{ij} is the i component of momentum that crosses a unit area perpendicular to the x^j direction. If we consider the interface between two infinitesimal cubes within the fluid, the force on that interface could have components perpendicular to and parallel to that interface. A perpendicular force gives rise to pressure, while a parallel force gives rise to shear. Shear is present in fluids with viscosity, but not in perfect fluids.

In the simple examples we've looked at, $T^{\alpha\beta}$ is symmetric, so that

$$T^{\alpha\beta}=T^{\beta\alpha} \tag{1}$$

This turns out to be true in general. The proof relies on physical concepts rather than mathematics.

First, we can look at the components T^{0i} and T^{i0}. T^{0i} is the energy per unit time that crosses a unit area perpendicular to the x^i direction. In relativity, mass and energy are equivalent, so T^{0i} is the mass per unit time that flows across a unit area. This amount of mass is the density ρ times the velocity component in the x^i direction, which is v^i. But ρv^i is the density of momentum in the x^i direction, so T^{0i} can also be regarded as the momentum density in the x^i direction, which is how we defined T^{i0} above. Therefore

$$T^{0i} = T^{i0} \tag{2}$$

Proving that $T^{ij} = T^{ji}$ is a bit more subtle. The usual argument considers an infinitesimal cube of side length s. We consider the forces on the sides of the cube. To make matters simpler, we consider the fluid in a frame that is momentarily moving with the fluid element, which Schutz calls the 'momentarily comoving reference frame' or MCRF. The point is that if we can show that a tensor is symmetric in one reference frame, then due to the frame-independence of tensor equations, it must be true in all frames.

Consider first the two faces of the cube that are perpendicular to the x axis, and let the first face be at $x = -\frac{s}{2}$ and the other face be at $x = \frac{s}{2}$. The force on the $x = -\frac{s}{2}$ face due to another infinitesimal cube that shares this face is determined by the components T^{ix}, that is, by the flow of momentum in the x^i direction that crosses the face perpendicular to $x = x^1$. In a general fluid, none of the T^{ix} is zero, since we could have both pressure and shear forces.

Since T^{ix} is the rate of momentum flowing across a surface of constant time, it is in effect the rate of change of momentum density. From Newton's law $F = \frac{dp}{dt}$, the rate of change of momentum is the force, so T^{ix} is the force density or force per unit area. Thus the force on $x = -\frac{s}{2}$ face is the force density times the area of the face, which is s^2. This force has (potentially) 3 components, which are given by

$$F^i_{x=-\frac{s}{2}} = T^{ix}_{x=-\frac{s}{2}} s^2 \tag{3}$$

On the $x = \frac{s}{2}$ face, the force is

$$F^i_{x=\frac{s}{2}} = T^{ix}_{x=\frac{s}{2}} s^2 \tag{4}$$

Since we're in the MCRF, the total force on the fluid element must be zero, at least in the limit as $s \to 0$. If the force were not zero, then as the volume and hence the mass becomes infinitesimal, a non-zero force would give rise to an infinite acceleration. Thus we have the condition that the two forces on opposite sides of the cube must approximately cancel, so

$$F^i_{x=-\frac{s}{2}} \approx -F^i_{x=\frac{s}{2}} \tag{5}$$

We can do a similar argument for forces on the two sides of the cube perpendicular to the $x^2 = y$ axis and to the $x^3 = z$ axis, from which we find

$$\begin{aligned} F^i_{y=-\frac{s}{2}} &= T^{iy}_{y=-\frac{s}{2}} s^2 \\ F^i_{y=\frac{s}{2}} &= T^{iy}_{y=\frac{s}{2}} s^2 \\ F^i_{z=-\frac{s}{2}} &= T^{iz}_{z=-\frac{s}{2}} s^2 \\ F^i_{z=\frac{s}{2}} &= T^{iz}_{z=\frac{s}{2}} s^2 \end{aligned} \tag{6}$$

The shear forces due to T^{ij} with $i \neq j$ give rise to a torque about the z axis. The component of this torque due to forces on the $x = -\frac{s}{2}$ and $x = \frac{s}{2}$ faces is found from the z component of $\boldsymbol{r} \times \boldsymbol{F}_x$, where the vector $\boldsymbol{r}$ points from the centre of the cube to the midpoint of the corresponding face. Here, we are making the assumption that the cube is small enough that the force is constant over each face, so we can approximate the torque by concentrating the force at the midpoint of each face.

For the $x = -\frac{s}{2}$ face, $\boldsymbol{r} = \left[-\frac{s}{2}, 0, 0\right]$ so

$$\left(\boldsymbol{r} \times \boldsymbol{F}_x\right)_z = -\frac{s}{2} F^y_x \tag{7}$$

$$= -\frac{s}{2} T^{yx}_{x=-\frac{s}{2}} s^2 \tag{8}$$

$$= -\frac{s^3}{2} T^{yx}_{x=-\frac{s}{2}} \tag{9}$$

At the $x = +\frac{s}{2}$ face, $\boldsymbol{r} = \left[\frac{s}{2}, 0, 0\right]$. Using the approximation 5, $\boldsymbol{F}^i_{x=\frac{s}{2}} \approx -\boldsymbol{F}^i_{x=-\frac{s}{2}}$. Thus on this face

$$\left(\boldsymbol{r} \times \boldsymbol{F}_x\right)_z = \frac{s}{2}\left(-F^y_x\right) \tag{10}$$

$$= -\frac{s}{2} T^{yx}_{x=-\frac{s}{2}} s^2 \tag{11}$$

$$= -\frac{s^3}{2} T^{yx}_{x=-\frac{s}{2}} \tag{12}$$

That is, the z component of torque due to the forces on the opposite faces is the same on both faces.

We can do the same calculation for the y faces. For $y = -\frac{s}{2}$, $\boldsymbol{r} = \left[0, -\frac{s}{2}, 0\right]$ and the z component of the torque is

$$(\boldsymbol{r}\times\boldsymbol{F}_y)_z = -\left(-\frac{s}{2}\right)F_y^x \tag{13}$$
$$= \frac{s}{2}T^{xy}_{y=-\frac{s}{2}}s^2 \tag{14}$$
$$= \frac{s^3}{2}T^{xy}_{y=-\frac{s}{2}} \tag{15}$$

As with the x faces, the torque due to the force on the $y=\frac{s}{2}$ face is the same as that from the $y=-\frac{s}{2}$ face.

With the approximation 5, we can drop the distinction between the two opposite faces in each case, and we have for each face:

$$\begin{aligned}(\boldsymbol{r}\times\boldsymbol{F}_x)_z &= -\frac{s^3}{2}T^{yx}\\ (\boldsymbol{r}\times\boldsymbol{F}_y)_z &= \frac{s^3}{2}T^{xy}\end{aligned} \tag{16}$$

What about the faces perpendicular to the z axis? It seems that there could be a contribution to the torque here due to the shear forces on these faces. However, with the approximations we're making that the forces are concentrated at the midpoints of the faces, we have, for the face at $z=-\frac{s}{2}$, $\boldsymbol{r}=\left[0,0,-\frac{s}{2}\right]$, so the z component of $\boldsymbol{r}\times\boldsymbol{F}_z$ is zero. Therefore the z faces make no contribution to the torque.

Combining the torques above, we find the total torque is

$$\boldsymbol{T} = 2(\boldsymbol{r}\times\boldsymbol{F}_x)_z + 2(\boldsymbol{r}\times\boldsymbol{F}_y)_z \tag{17}$$
$$= s^3\left(T^{xy}-T^{yx}\right) \tag{18}$$

The angular analog of Newton's law is that the torque equals moment of inertia I times angular acceleration. If the angular velocity of the cube is $\boldsymbol{\omega}$, then we have

$$\boldsymbol{T} = I\dot{\boldsymbol{\omega}} \tag{19}$$

The moment of inertia of a cube about an axis through its centre is

$$I = \frac{1}{6}ms^2 \tag{20}$$
$$= \frac{1}{6}\rho s^5 \tag{21}$$

where the mass of the cube is its density ρ times its volume s^3. Therefore, the angular equation of motion 19 is

$$\boldsymbol{T} = \frac{1}{6}\rho s^5 \dot{\boldsymbol{\omega}} \tag{22}$$

Equating this with 18 and considering the z component so we can deal with magnitudes rather than vectors, we have

$$s^3 \left(T^{xy} - T^{yx}\right) = \frac{1}{6}\rho s^5 \dot{\omega} \tag{23}$$

which gives an angular acceleration of

$$\dot{\omega} = \frac{6\left(T^{xy} - T^{yx}\right)}{\rho s^2} \tag{24}$$

The point is that, as $s \to 0$, the angular acceleration becomes infinite unless $T^{xy} = T^{yx}$. Since a physical fluid cannot support infinite vortices, we conclude

$$T^{xy} = T^{yx} \tag{25}$$

We could do the same analysis for the torques about the other two axes, so the general conclusion is that the stress-energy tensor is fully symmetric:

$$\boxed{T^{\alpha\beta} = T^{\beta\alpha}} \tag{26}$$