STRESS-ENERGY TENSOR IN THE WEAK FIELD LIMIT

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In the weak field limit, the Ricci tensor becomes

$$R_{jm}=\frac{1}{2}\left(\partial_{j}H_{m}+\partial_{m}H_{j}-\eta^{nl}\partial_{n}\partial_{l}h_{jm}\right) \tag{1}$$

where

$$H_{m}\equiv\eta^{nl}\left(\partial_{n}h_{lm}-\frac{1}{2}\partial_{m}h_{nl}\right) \tag{2}$$

and the h_{ij} is the perturbation on the flat metric, so that

$$g_{ij}=\eta_{ij}+h_{ij} \tag{3}$$

Because we can introduce a coordinate transformation for the four coordinates in the form

$$\left(x'\right)^{i}=f^{i}\left(x^{j}\right) \tag{4}$$

there are four degrees of freedom that we can play with in specifying the form of H_i. It turns out (we may get around to a proof in some future post) that it is always possible to find a coordinate system in which all $H_i=0$. If we use such a coordinate system then the Ricci tensor is

$$R_{jm}=-\frac{1}{2}\eta^{nl}\partial_{n}\partial_{l}h_{jm} \tag{5}$$

so the Einstein equation becomes

$$-\frac{1}{2}\eta^{nl}\partial_{n}\partial_{l}h_{jm}=8\pi G\left(T_{jm}-\frac{1}{2}\eta_{jm}T\right) \tag{6}$$

Introducing the d'Alembertian operator

$$\square^{2}\equiv\eta^{nl}\partial_{n}\partial_{l}=-\frac{d^{2}}{dt^{2}}+\nabla^{2} \tag{7}$$

we can write the Einstein equation as

$$-\frac{1}{2}\square^{2}h_{jm}=8\pi G\left(T_{jm}-\frac{1}{2}\eta_{jm}T\right) \tag{8}$$

As it stands, this equation is a set of uncoupled differential equations for the h_{jm} so in principle they can be solved. However, we can invoke another approximation by assuming that the system is in a steady state so that all time derivatives are zero. This doesn't necessarily mean that the masses are all stationary, since we might have a star rotating at a constant angular velocity. In such cases, the stress-energy tensor T_{jm} is constant in time so we would expect h_{jm} to be independent of time as well. In that case, we get

$$\nabla^2 h_{jm} = -16\pi G\left(T_{jm} - \frac{1}{2}\eta_{jm}T\right) \tag{9}$$

This equation should look familiar from electrodynamics, where it is formally equivalent to Poisson's equation for the electrostatic potential in terms of the charge distribution. In that case we had

$$\nabla^2 V = -\nabla\cdot\mathbf{E} = -\frac{\rho}{\epsilon_0} \tag{10}$$

where ρ here is the charge density. The solution of this equation is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho\left(\mathbf{r}'\right)d^3\mathbf{r}' \tag{11}$$

By replacing ρ/ϵ_0 with $16\pi G\left(T_{jm} - \frac{1}{2}\eta_{jm}T\right)$ we can find h_{jm} as

$$h_{jm} = 2G\int\frac{2T_{jm} - \eta_{jm}T}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{12}$$

Thus if we know the stress-energy tensor as a function of position, we can work out the perturbations on the flat metric h_{jm}.

We can look at this equation for the case of a perfect fluid, where the stress-energy tensor is

$$T_{ij} = (\rho_0 + P_0)u_i u_j + P_0 g_{ij} \tag{13}$$

where ρ_0 and P_0 are the fluid's density and pressure in its rest frame, and u_i is the fluid's four-velocity in the observer's frame. We need to find the numerator of the integrand in 12 for each component. First, we work out the stress-energy scalar T:

$$T = g^{ij}T_{ij} \tag{14}$$
$$= (\rho_0 + P_0)g^{ij}u_i u_j + P_0 g^{ij}g_{ij} \tag{15}$$
$$= -(\rho_0 + P_0) + 4P_0 \tag{16}$$
$$= -\rho_0 + 3P_0 \tag{17}$$

since $g^{ij}u_iu_j = -1$ and $g^{ij}g_{ij} = \delta^i_i = 4$. Since T is a scalar, this result is valid in all coordinate systems. Also, we haven't yet used any approximations, so this result is valid for all perfect fluids, even ones where the density and pressure are large.

Now let's assume that ρ_0 and P_0 are small and so we keep only up to first order terms, so any product of ρ_0 or P_0 with h^{ij} can be ignored. Thus

$$T_{ij} - \frac{1}{2}g_{ij}T \approx T_{ij} - \frac{1}{2}\eta_{ij}T \tag{18}$$

$$2T_{ij} - g_{ij}T \approx 2T_{ij} - \eta_{ij}T \tag{19}$$

$$\approx 2(\rho_0 + P_0)u_iu_j + 2P_0\eta_{ij} - \eta_{ij}(-\rho_0 + 3P_0) \tag{20}$$

$$= 2(\rho_0 + P_0)u_iu_j + \eta_{ij}(\rho_0 - P_0) \tag{21}$$

There are 3 cases to consider. First, $i = j = t$ and use $\eta_{tt} = -1$. Further, we'll assume that the spatial velocity components are all small, so $u_t \approx -1$ and $u_iu_j \approx 0$ if i and j are both spatial indices.

$$2T_{tt} - \eta_{tt}T = 2(\rho_0 + P_0)u_tu_t + \eta_{tt}(\rho_0 - P_0) \tag{22}$$

$$= 2(\rho_0 + P_0) - (\rho_0 - P_0) \tag{23}$$

$$= \rho_0 + 3P_0 \tag{24}$$

Now suppose $i = t$ and j is a spatial index (or vice versa; since everything is symmetric it makes no difference). Then

$$2T_{tj} - \eta_{tj}T = 2T_{tj} \tag{25}$$

$$= 2(\rho_0 + P_0)u_tu_j \tag{26}$$

$$= -2(\rho_0 + P_0)u_j \tag{27}$$

Finally, if both i and j are spatial indices we get, if $i \neq j$

$$2T_{ij} - g_{ij}T = 2(\rho_0 + P_0)u_iu_j + \eta_{ij}(\rho_0 - P_0) \tag{28}$$

$$\approx 0 \tag{29}$$

since the first term involves the second order term u_iu_j and in the second term $\eta_{ij} = 0$ if $i \neq j$.

Now if $i = j$ we get

$$2T_{ii} - g_{ii}T = 2(\rho_0 + P_0)u_iu_i + \eta_{ii}(\rho_0 - P_0) \tag{30}$$

$$\approx \rho_0 - P_0 \tag{31}$$

again, since the first term has the second order factor u_i^2 and in the second term $\eta_{ii}=+1$ if i is a spatial index.

Therefore, if we know ρ_0 and P_0 as functions of position we can work out the perturbations h_{ij} to the metric.

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