TENSOR EQUATIONS ARE VALID IN ALL COORDINATES

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One feature of tensors that is often useful in deriving tensor equations is that if a tensor equation can be shown to be valid in one coordinate system, then it is valid in all other coordinate systems. In many cases, it's easier to derive a tensor equation in a locally flat coordinate system, and then to use the universally valid nature of tensor equations to state that the equation is valid in all coordinate systems.

This property can be derived from the fact that if *all* components of a tensor are zero in on coordinate system, then they are zero in all coordinate systems. To see why this is true, we consider the transformation equations for the components of both contravariant and covariant tensors. For contravariant tensors, the conversion from one coordinate system $\{x^{\mu}\}$ to another $\{y^{\nu}\}$ is done by multiplying each contravariant index by the derivative $\partial y^{\nu}/\partial x^{\mu}$ and summing. That is, for a rank-2 tensor $T^{\mu\nu}$, we have

$$T^{\alpha\beta}(y)=\frac{\partial y^{\alpha}}{\partial x^{\mu}}\frac{\partial y^{\beta}}{\partial x^{\nu}}T^{\mu\nu}(x) \tag{1}$$

Contravariant tensors of different rank have one derivative for each index. The notation $T^{\alpha\beta}(y)$ means that the tensor T's components are written in the $\{y\}$ coordinate system.

For a covariant tensor, the transformation is similar, except that the derivatives are inverted. That is, for a rank-2 covariant tensor $C_{\mu\nu}$, we have

$$C_{\alpha\beta}(y)=\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}C_{\mu\nu}(x) \tag{2}$$

Tensors with mixed contravariant and covariant indices have a derivative of the corresponding type for each index. So for example

$$M^{\alpha}{}_{\beta}(y)=\frac{\partial y^{\alpha}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial y^{\beta}}M^{\mu}{}_{\nu}(x) \tag{3}$$

The important thing to note in all these transformation formulas is that each tensor component in the $\{y\}$ system is given by a linear combination of the components in the $\{x\}$ system, with the coefficients being the derivatives corresponding to the contravariant or covariant nature of each index. From this, we see that if *all* the components of a tensor such as $T^{\mu\nu}(x)$ are zero in the $\{x\}$ system, then *all* the components of T will be zero in *all* coordinate systems, since the components of T in any system are linear combinations of a bunch of zero terms.

Having established this, we can generalize the statement to the property that a tensor *equation* valid in one coordinate system must be true in *all* coordinate systems. Suppose we have the equation

$$T^{\mu\nu}(x) = U^{\mu\nu}(x) \tag{4}$$

which is valid in the $\{x\}$ system. We subtract $U^{\mu\nu}$ from both sides to get

$$T^{\mu\nu}(x) - U^{\mu\nu}(x) = 0 \tag{5}$$

Since a linear combination of two tensors of the same type is always another tensor of that type, we see that the tensor defined by

$$W^{\mu\nu}(x) \equiv T^{\mu\nu}(x) - U^{\mu\nu}(x) = 0 \tag{6}$$

has all its components equal to zero in the $\{x\}$ system, and thus they must be zero in all systems. From this it follows that 4 is valid in all coordinate systems.

One point must be stressed here. The above discussion applies only if the objects with which we're dealing are really tensors. That is, it's *not* true that if the components of some arbitrary object that is not a tensor are all zero in one coordinate system, then they are zero in all systems. A good example of this is the Christoffel symbols.

For example, in rectangular coordinates in flat 2-d space, the Christoffel symbols are all zero. However, if we use polar coordinates r and θ, the Christoffel symbols are

$$\Gamma^{r}_{\ ij} = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} \tag{7}$$

$$\Gamma^{\theta}_{\ ij} = \begin{bmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{bmatrix} \tag{8}$$

Here, the index i is the row index and j is the column index, each of which can be r or θ.

In this case, the space we're considering is flat 2-dim space in both cases, so the validity of a tensor equation as described above must apply to all coordinate systems used to describe this space. The fact that the Christoffel symbols are all zero for one coordinate system (rectangular) but not all zero for another system (polar) shows that they cannot be tensors. It is only when they are included in the formula for the covariant derivative that the combination is a tensor. That is, for a vector V^{α}, the covariant derivative

$$V^{\alpha}{}_{;\beta}=V^{\alpha}{}_{,\beta}+V^{\gamma}\Gamma^{\alpha}{}_{\gamma\beta} \tag{9}$$

is a tensor, even though each term on the RHS is, on its own, not a tensor.

Thus if we can show that the covariant derivative is equal to some tensor in one coordinate system, then that equation is also valid in all systems. However, deriving the ordinary derivative $V^{\alpha}{}_{,\beta}$ or the Christoffel symbol $\Gamma^{\alpha}{}_{\gamma\beta}$ in one system doesn't, in general, tell you anything about its value in another system.

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