

## TENSORS AND ONE-FORMS

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### 1. TENSORS

Tensors feature prominently in relativity, so it's a good idea to get a feel for just what they are. First, it's important to realize that a tensor, like a vector, is an object that is independent of the coordinate system being used. What *do* depend on the coordinate system are the components of the tensor.

A tensor (written as  $\binom{0}{N}$  by Schutz in *A First Course in General Relativity*) is actually a linear function of  $N$  vectors. The result of this function is to produce an ordinary number after operating on the  $N$  vectors. Thus the metric tensor  $g$  is actually a function of 2 vectors, with this function providing the scalar product of the two vectors. That is,

$$g(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B} \quad (1)$$

We're usually used to seeing the scalar product written using the *components* of the metric tensor in a given coordinate system. In a Lorentz frame, for example, we have

$$\vec{A} \cdot \vec{B} = \eta_{\mu\nu} A^\mu B^\nu \quad (2)$$

The scalar product is independent of the coordinate system (it's the same in every Lorentz frame), and is a function of two vectors, so the metric tensor defined this way is a  $\binom{0}{2}$  tensor.

### 2. ONE-FORMS

The simplest tensor is a  $\binom{0}{0}$  tensor, which is a function of no vectors, and just returns a scalar. Next up the scale we have a  $\binom{0}{1}$  tensor, called a *one-form* or 1-form. According to the definition above, a one-form is a function of a single vector and returns a scalar as its result.

Consider the momentum  $\mathbf{p}$  of a particle. This is a vector (we're dealing with relativity, so all vectors are four-vectors). According to quantum mechanics, however, a free particle with momentum  $\mathbf{p}$  is represented by a plane wave. This wave has evenly spaced peaks and troughs which are described

by the phase  $\phi$  of the wave at any point. For a given value of the phase  $\phi$ , we can draw a surface in 4-space of all locations where the phase has that value. The pattern of such surfaces is an example of a one-form.

Now consider two events  $\mathcal{P}_0$  and  $\mathcal{P}$  and draw the vector between these two points as

$$\mathbf{v} = \mathcal{P} - \mathcal{P}_0 \quad (3)$$

This vector will pierce some number of these constant surfaces represented by the one-form. If  $\mathbf{v}$  connects two points where the phase is equal, then it pierces zero surfaces; if it connects two points with different phases, then the number of surfaces pierced is the difference in phase between these two points.

When we're considering a particle's momentum, the one-form representing the phase is often represented by the symbol  $\tilde{\mathbf{k}}$ . The number of surfaces of  $\tilde{\mathbf{k}}$  pierced by  $\mathbf{v}$  is then written

$$\langle \tilde{\mathbf{k}}, \mathbf{v} \rangle \quad (4)$$

Schutz uses the notation

$$\tilde{k}(\mathbf{v}) \quad (5)$$

for the same thing. Schutz's notation is perhaps clearer, since it emphasizes the fact that a one-form (a tensor, remember) is a function of a single vector.

### 3. COMPONENTS

When dealing with vectors, we usually have to resort to some component representation in a particular coordinate system. Schutz gives us a method for determining the components of a one-form.

In a given coordinate system  $\mathcal{O}$ , we have a set of basis vectors  $\vec{e}_\alpha$ . The components of a one-form  $\tilde{p}$  are obtained by letting it act on each basis vector. That is

$$p_\alpha \equiv \tilde{p}(\vec{e}_\alpha) \quad (6)$$

From this, we find the expression for the one-form acting on a general vector  $\vec{A}$ . Since  $\vec{A}$  is written in terms of the basis vectors as

$$\vec{A} = A^\alpha \vec{e}_\alpha \quad (7)$$

and the one-form is a linear function, we have, since the components  $A^\alpha$  are just numbers:

$$\tilde{p}(\vec{A}) = \tilde{p}(A^\alpha \vec{e}_\alpha) \quad (8)$$

$$= A^\alpha \tilde{p}(\vec{e}_\alpha) \quad (9)$$

$$= A^\alpha p_\alpha \quad (10)$$

We can use this result to obtain a set of basis one-forms in a given frame  $\mathcal{O}$ . We would like to be able to write a one-form in terms of a set  $\tilde{\omega}^\alpha$  of basis one-forms as

$$\tilde{p} = p_\alpha \tilde{\omega}^\alpha \quad (11)$$

As it stands, this is an abstract equation, since we must remember that a one-form is a function of a vector. Thus in order for 11 to make sense, we have to use it to act on a vector. Let's applying to the vector  $\vec{A}$  above. We get

$$\tilde{p}(\vec{A}) = p_\alpha \tilde{\omega}^\alpha(\vec{A}) \quad (12)$$

$$= p_\alpha \tilde{\omega}^\alpha(A^\beta \vec{e}_\beta) \quad (13)$$

$$= p_\alpha A^\beta \tilde{\omega}^\alpha(\vec{e}_\beta) \quad (14)$$

Notice that this formula involves two sums; one over  $\alpha$  and one over  $\beta$ . Also note that we have a set of 4  $\tilde{\omega}^\alpha$ s, each of which acts on each of the 4 basis vectors  $\vec{e}_\beta$ . Each one-form has 4 components, as given by 6. Now in order for 14 to be consistent with 10 for any vector  $\vec{A}$ , we must have

$$\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta \quad (15)$$

That is, the components of  $\tilde{\omega}^\alpha$  in the coordinate system  $\mathcal{O}$  must be

$$\begin{aligned} \tilde{\omega}^0 &\rightarrow (1, 0, 0, 0) \\ \tilde{\omega}^1 &\rightarrow (0, 1, 0, 0) \\ \tilde{\omega}^2 &\rightarrow (0, 0, 1, 0) \\ \tilde{\omega}^3 &\rightarrow (0, 0, 0, 1) \end{aligned} \quad (16)$$

These are the basis one-forms relative to the basis vectors  $\vec{e}_\alpha$ . If we used different basis vectors, we'd get different basis one-forms.

What is the relation between a vector  $\mathbf{p}$  and a corresponding one-form  $\tilde{\mathbf{p}}$ ? In quantum mechanics, the phase becomes the momentum when it is multiplied by  $\hbar$  (in relativistic quantum theory, we usually take  $\hbar = 1$ , so then momentum and phase become the same). If we multiply  $\tilde{\mathbf{k}}$  above by  $\hbar$ ,

we then get the momentum one-form  $\tilde{\mathbf{p}}$ . If we now use this one-form to act on some vector  $\mathbf{v}$ , we get

$$\langle \tilde{\mathbf{p}}, \mathbf{v} \rangle \quad (17)$$

which is the number of surfaces of  $\tilde{\mathbf{p}}$  pierced by  $\mathbf{v}$ . This is just another way of saying that we are finding the projection of  $\mathbf{v}$  onto the vector  $\mathbf{p}$ , which is given by the scalar product. In other words,

$$\mathbf{p} \cdot \mathbf{v} = \langle \tilde{\mathbf{p}}, \mathbf{v} \rangle \quad (18)$$

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