# TORUS METRIC

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Post date: 10 Feb 2023.

To find the metric on the surface of a torus, we can first consider what coordinates we need to specify a position on a torus. See Figure 1. The top view shows the torus lying flat on the $xy$ plane. The radius of the torus (from its centre to halfway through the tube) is $L$ and the angle measured counterclockwise from the $x$ axis is $\phi$, so $0 \le \phi < 2\pi$.

The inset figure at the bottom right shows a vertical cross-section through the tube of the torus. The radius of the tube is $a$ and the angle measured counterclockwise from the top of the torus is $\theta$, so $0 \le \theta < 2\pi$. Thus the radius of the torus from its centre to the inner ring is $L-a$ and to the outer ring is $L+a$.

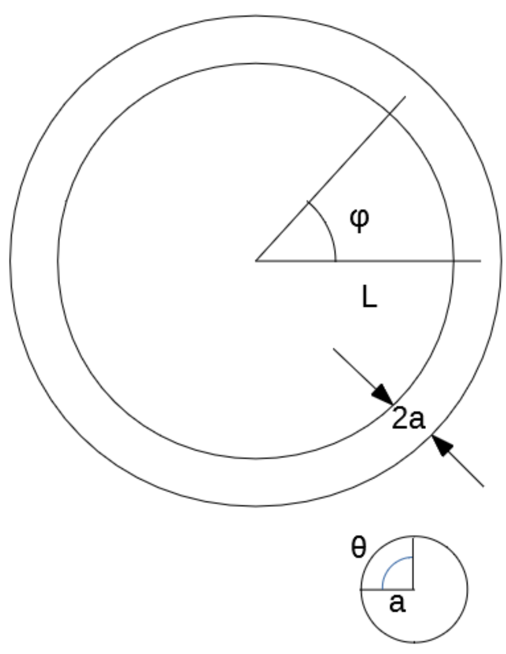


FIGURE 1. Torus. Horizontal view (top) and cross section of tube (bottom).

The $z$ (vertical) coordinate of a point on the torus is therefore determined entirely by the vertical cross section, and is

$$z = a\cos\theta \tag{1}$$

so that $z = 0$ in the horizontal plane that bisects the torus horizontally (the $xy$ plane), where $\theta = \frac{\pi}{2}$.

The radial distance from the centre to a point on the torus is

$$r = L - a\sin\theta \tag{2}$$

so the $x$ and $y$ coordinates are

$$x = r\cos\phi = (L - a\sin\theta)\cos\phi \tag{3}$$
$$y = r\sin\phi = (L - a\sin\theta)\sin\phi \tag{4}$$

The infinitesimal increments are then

$$dx = -a\cos\theta\cos\phi d\theta - (L - a\sin\theta)\sin\phi d\phi \tag{5}$$
$$dy = -a\cos\theta\sin\phi d\theta + (L - a\sin\theta)\cos\phi d\phi \tag{6}$$
$$dz = -a\sin\theta d\theta \tag{7}$$

Squaring and adding gives (using Maple to do the algebra and trig simplifications)

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{8}$$
$$= a^2 d\theta^2 + \left(L^2 + a^2 - a^2\cos^2\theta - 2La\sin\theta\right) d\phi^2 \tag{9}$$
$$= a^2 d\theta^2 + (L - a\sin\theta)^2 d\phi^2 \tag{10}$$

Thus the metric is

$$g_{\mu\nu} = \left[\begin{array}{cc} a^2 & 0 \\ 0 & (L - a\sin\theta)^2 \end{array}\right] \tag{11}$$

My answer differs from Zee's because he uses $\theta$ that starts from the bottom of the tube rather than the top so I get a factor of $(L - a\sin\theta)$. I was too lazy to go back and change everything.