TWIN PARADOX

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The so-called 'twin paradox' in special relativity consists of a pair of twins, say Andy and Betty, one of whom (Betty) takes off in a relativistic rocket while her brother remains on Earth. After travelling to some distant star, Betty turns around and returns to Earth. Due to time dilation, Betty is now younger than Andy when she arrives back on Earth.

The twin paradox can be analyzed using Lorentz transformations to get an idea of what each twin perceives at key points in the journey. Let Andy's frame be unprimed (x,t) and Betty's frame be primed (x',t'). At $t=t'=0$ (which corresponds to the twins' 21st birthday) and $x=x'=0$ in both systems, Betty leaves on a spaceship travelling at $v=\frac{4}{5}$ heading towards a star X. Upon arriving at X, Betty immediately transfers to another spaceship and heads back to Earth, also at $v=\frac{4}{5}$, arriving on her 39th birthday (according to her own reckoning, so to Betty, the round trip took $\Delta t'=18$ years). This speed gives a relativistic factor of

$$\gamma=\frac{1}{\sqrt{1-v^2}}=\frac{5}{3} \tag{1}$$

Andy sees Betty as moving in both directions, so to him, Betty's clock runs slow. Using the time dilation formula, Andy sees a time interval of

$$\begin{aligned}
\Delta t &= \gamma\Delta t' && (2)\\
&= \frac{5}{3}\times 18 && (3)\\
&= 30 \text{ years} && (4)
\end{aligned}$$

Thus Andy will be $21+30=51$ years old when Betty arrives home.

According to Andy, the two legs of Betty's trip took the same time, 15 years, at a speed of $\frac{4}{5}$, so star X is at a distance of

$$d_X=\frac{4}{5}\times 15=12 \text{ light years} \tag{5}$$

In Andy's frame, the coordinates of Betty's jump between spaceships is

$$(x,t) = (12,15) \tag{6}$$

In Betty's outgoing frame, we can apply Lorentz transformations to get her coordinates

$$\begin{aligned}
x' &= \gamma(x - vt) && (7)\\
&= \frac{5}{3}\left(12 - \frac{4}{5}15\right) && (8)\\
&= 0 && (9)\\
t' &= \gamma(t - xv) && (10)\\
&= \frac{5}{3}\left(15 - \frac{4}{5}12\right) && (11)\\
&= 9\text{ years} && (12)
\end{aligned}$$

These values are consistent, since Betty doesn't move relative to her own frame, so we'd expect $x' = 0$, and since her clock runs slow relative to Andy's clock, the time interval is shortened by a factor of $1/\gamma$ giving $t' = 9$ years.

Now consider the frame (indicated by a double prime $\mathcal{S}''$) fixed to Betty's returning spaceship. We'll fix the coordinates so that $t = t' = t'' = 0$ and $x = x' = x'' = 0$. Its velocity relative to Andy is $-v = -\frac{4}{5}c$, so in the $\mathcal{S}''$ frame, the jump occurs at

$$\begin{aligned}
x'' &= \gamma(x + vt) && (13)\\
&= \frac{5}{3}\left(12 + \frac{4}{5}15\right) && (14)\\
&= 40\text{ light years} && (15)\\
t'' &= \gamma(t + xv) && (16)\\
&= \frac{5}{3}\left(15 + \frac{4}{5}12\right) && (17)\\
&= 41\text{ years} && (18)
\end{aligned}$$

Thus for Betty to adjust her clock so that it agreed with the $\mathcal{S}''$ frame, she would have to advance it by 32 years just after making the jump. In this frame, the coordinates of her arrival back on Earth are given by transforming Andy's coordinates of $(x,t) = (0,30)$, so we have

$$
\begin{aligned}
x'' &= \gamma(x+vt) && (19)\\
&= \frac{5}{3}\left(0+\frac{4}{5}30\right) && (20)\\
&= 40 \text{ light years} && (21)\\
t'' &= \gamma(t+xv) && (22)\\
&= \frac{5}{3}(30+0) && (23)\\
&= 50 \text{ years} && (24)
\end{aligned}
$$

Note that x'' doesn't change as Betty travels home, again because this is her own rest frame. Also, the time interval for her to get home is again $50-41=9$ years in her own frame, the same as for the outbound journey.

Andy's age, as viewed by Betty, depends on which spaceship she is on. Just before making the jump she is in the $\mathcal{S}'$ frame, so according to her, the time back on Earth at that point is given by taking $t'=9$, $x'=x=0$ in the transformation

$$
\begin{aligned}
t' &= \gamma(t-xv) && (25)\\
9 &= \frac{5}{3}(t-0) && (26)\\
t &= \frac{27}{5}=5.4 \text{ years} && (27)
\end{aligned}
$$

That is, Betty thinks Andy is actually 3.6 years *younger* than she is when she reaches the star. This is because to her, it is Andy who is moving so *his* clock runs slow by a factor of $1/\gamma$ relative to her 9 years. How can we reconcile this with the fact that Andy thinks that Betty takes 15 years to get to the star, so according to him, he is actually 6 years *older* than Betty when she reaches the star? The confusion arises because of differences in simultaneity as perceived by Andy and Betty. Andy says that the two events (Andy on Earth at age $21+15=36$ years in frame $\mathcal{S}$, and Betty arriving at the star) are simultaneous, but Betty disagrees with this, saying that the two events (Andy on Earth at age $21+5.4=26.4$ years in frame $\mathcal{S}'$, and Betty arriving at the star) are simultaneous. In general, two observers can agree about two events being simultaneous only if these events happen at the same location in both systems. In this case, one event occurs on Earth and the other at the star, so they are not in the same place.

Just after the jump, Betty can now calculate Andy's age in the $\mathcal{S}''$ system by using $t''=41$ years and $x=0$:

$$t'' = \gamma(t+xv) \tag{28}$$

$$41 = \frac{5}{3}(t+0) \tag{29}$$

$$t = \frac{123}{5}=24.6\text{ years} \tag{30}$$

Thus Betty now thinks that Andy is $21+24.6=45.6$ years old. Andy's age, of course, hasn't changed as Betty jumps between ships; only her perception of it has changed.

In her own frame, Betty says the return trip to Earth takes 9 years, so by the same logic as in 27, she says that 5.4 years elapse for Andy, making him $45.6+5.4=51$ years old when she arrives home. Thus both Andy and Betty agree on Andy's age when she gets back.

The difference between the two twins is, of course, that Betty undergoes deceleration and then acceleration in the reverse direction when she jumps between ships at the star. A more realistic treatment would have Betty gradually decelerating as she approached the star and then gradually accelerating as she got onto the other ship for the journey home. During this process, Betty is continuously changing reference frames resulting in her 'seeing' Andy age rapidly (but continuously, rather than the jump in age from 26.4 to 45.6 that happens here) in the process. Time dilation applies consistently to both Andy and Betty: whichever twin is moving relative to the other does perceive the other's clock to be running slow. The 'paradox' arises because Betty isn't in the same inertial frame throughout her journey. It is the switch between frames that causes the ages to get out of sync with each other. Thus the paradox is real, in the sense that different amounts of time actually do pass for the two twins.

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