VECTORS AND THE METRIC TENSOR

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We can define a general vector $\mathbf{A}$ in terms of the basis vectors $\mathbf{e}_i$ in a given coordinate system:

$$\mathbf{A} \equiv A^i \mathbf{e}_i \tag{1}$$

This is analogous to the definition of the infinitesimal displacement that we met earlier: $d\mathbf{s} = dx^i \mathbf{e}_i$. This has a couple of consequences. First, since the basis vectors are not necessarily either unit vectors or orthogonal, this definition may be different from the usual definition of a vector that you're used to from linear algebra courses.

Second, we require the transformation of a vector's components between coordinate systems to be the same as the components of $d\mathbf{s}$, which means that

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \tag{2}$$

Finally, the square of a vector follows the same pattern as the square of the increment ds^2:

$$A^2 = \mathbf{A} \cdot \mathbf{A} = g_{ij} A^i A^j \tag{3}$$

As an example, consider the case of uniform circular motion. From elementary physics, we know that, in polar coordinates, the radial component of the velocity $v^r = 0$ (since the object is always at the same distance from the origin) and the tangential component is v^θ. The metric for polar coordinates is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{4}$$

so we have

$$
\begin{aligned}
v^2 &= g_{ij} v^i v^j &(5)\\
&= 1\times 0\times 0 + r^2 \times v^\theta \times v^\theta &(6)\\
&= \left(v^\theta\right)^2 r^2 &(7)\\
v^\theta &= \frac{v}{r} &(8)
\end{aligned}
$$

Note that because we're not using unit basis vectors (that is, $|\boldsymbol{e}_\theta| = r \neq 1$ in general), the tangential component $v^\theta \neq v$, where v is the actual speed of the object around the circle.

To transform this vector to rectangular coordinates, we use the transformations

$$
\begin{aligned}
x &= r\cos\theta &(9)\\
y &= r\sin\theta &(10)\\
r &= \sqrt{x^2+y^2} &(11)\\
\theta &= \arctan\frac{y}{x} &(12)
\end{aligned}
$$

So we have

$$
v'^i = \frac{\partial x'^i}{\partial x^j} v^j \qquad (13)
$$

where the primed system is rectangular and the unprimed is polar. So

$$
\begin{aligned}
v^x &= \frac{\partial x}{\partial r} v^r + \frac{\partial x}{\partial \theta} v^\theta &(14)\\
&= -r\sin\theta \frac{v}{r} &(15)\\
&= -v\sin\theta &(16)\\
v^y &= \frac{\partial y}{\partial r} v^r + \frac{\partial y}{\partial \theta} v^\theta &(17)\\
&= r\cos\theta \frac{v}{r} &(18)\\
&= v\cos\theta &(19)
\end{aligned}
$$

The square is invariant, since using the rectangular metric

$$v^2 = g_{ij}v^iv^j \tag{20}$$
$$= (-v\sin\theta)^2 + (v\cos\theta)^2 \tag{21}$$
$$= v^2 \tag{22}$$

Now let's look at the inverse problem. This time we have an object moving at a constant speed v in the $+y$ direction, so that $v^x = 0$, $v^y = v$. To convert this to polar coordinates, we need the derivatives

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \tag{23}$$
$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \tag{24}$$
$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} \tag{25}$$
$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} \tag{26}$$

Then we get

$$v^r = \frac{y}{\sqrt{x^2+y^2}}v \tag{27}$$
$$= v\sin\theta \tag{28}$$
$$v^\theta = \frac{x}{x^2+y^2}v \tag{29}$$
$$= \frac{\cos\theta}{r}v \tag{30}$$

If the object starts at $(x,y) = (b,0)$ at $t = 0$, then $y(t) = vt$ and $x(t) = b$. In polar coordinates we get

$$r(t) = \sqrt{b^2+(vt)^2} \tag{31}$$
$$\theta(t) = \arctan\frac{vt}{b} \tag{32}$$
$$v^r = \frac{vt}{\sqrt{b^2+(vt)^2}}v \tag{33}$$
$$v^\theta = \frac{b}{b^2+(vt)^2}v \tag{34}$$

At $t = 0$, the motion is entirely in the θ direction, since the object is moving tangent to the circle $r = b$ at that time. As time increases, the motion gradually transfers over to the radial direction, with $\lim_{t\to\infty} v^r = v$.

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