VECTORS IN POLAR COORDINATES

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We'll consider coordinate transformations between rectangular and polar coordinates in two dimensions. Polar coordinates r and θ are defined in terms of rectangular coordinates by

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\frac{y}{x} \end{aligned} \tag{1}$$

and the inverse

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{2}$$

The radial coordinate r is the distance from the origin and the polar angle θ is measured counterclockwise from the horizontal axis.

Displacements in r and θ can be related to displacements in x and y using the usual rule for derivatives:

$$\begin{aligned} \Delta r &= \frac{\partial r}{\partial x}\Delta x + \frac{\partial r}{\partial y}\Delta y \\ \Delta\theta &= \frac{\partial \theta}{\partial x}\Delta x + \frac{\partial \theta}{\partial y}\Delta y \end{aligned} \tag{3}$$

The transformation can be written in matrix form with the transformation matrix

$$\Lambda^{\alpha'}{}_{\beta} = \left[\begin{array}{cc} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{array}\right] \tag{4}$$

where the primed index α' refers to a polar coordinate and the unprimed index β to a rectangular coordinate.

Although Schutz uses the same symbol Λ here as for Lorentz transformations, the Λ matrix here is just a coordinate transformation and is not a Lorentz transformation.

We can work out the derivatives from 1 and we have

$$\Lambda^{\alpha'}{}_{\beta} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \tag{7}$$

Using this matrix, the transformation rule for an arbitrary vector $\vec{V}$ is

$$V^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} V^{\beta} \tag{8}$$

Example 1. Consider the vector defined in rectangular coordinates as

$$\vec{W} = [1,1] \tag{9}$$

To transform this to polar coordinates, we use 8, and we have

$$\vec{W}_{\text{polar}}(r,\theta) = \left[\cos\theta + \sin\theta, \frac{1}{r}(-\sin\theta + \cos\theta)\right] \tag{10}$$

This result may seem surprising, since the original definition 9 of $\vec{W}$is a fixed vector with constant components. How could its representation 10 in polar coordinates depend on r and θ?

The key point is that in rectangular coordinates, the basis vectors $\vec{e}_x$ and $\vec{e}_y$ are constant over the entire $x-y$ plane, so the representation of $\vec{W}$ can be written as

$$\vec{W} = (1)\vec{e}_x + (1)\vec{e}_y \tag{11}$$

and the coefficients of the basis vectors are always $[1,1]$.

In polar coordinates, the basis vectors $\vec{e}_r$ and $\vec{e}_\theta$ are *not* constant; rather they vary depending on which point in the plane we are at. The basis vector $\vec{e}_r$ always points radially away from the origin, and $\vec{e}_\theta$ is always perpendicular to $\vec{e}_r$ at any given point. The polar basis vectors are given by Schutz's eqns 5.22 and 5.23 as

$$\begin{aligned} \vec{e}_r &= \cos\theta \vec{e}_x + \sin\theta \vec{e}_y \\ \vec{e}_\theta &= -r\sin\theta \vec{e}_x + r\cos\theta \vec{e}_y \end{aligned} \tag{12}$$

From this we see explicitly that the polar basis vectors vary with their position, and in the case of $\vec{e}_\theta$, it is not even a unit vector, having magnitude r.

We can see how this applies to 10. From the rectangular form 9 we see that the magnitude of $\vec{W}$ is $\sqrt{2}$ and its angle with respect to the x axis is $\frac{\pi}{4}$, so try $r=\sqrt{2}$ and $\theta=\frac{\pi}{4}$ in 10. This gives us

$$\vec{W}\left(\sqrt{2},\frac{\pi}{4}\right) = \left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2},\frac{1}{\sqrt{2}}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)\right] \tag{13}$$

$$= \left[\sqrt{2},0\right] \tag{14}$$

In other words, at this point,

$$\vec{W}\left(\sqrt{2},\frac{\pi}{4}\right) = \sqrt{2}\vec{e}_r \tag{15}$$

This makes sense, as $\vec{W}$ is a radial vector with length $\sqrt{2}$.

Now suppose we try $r=1,\theta=0$. In this case

$$\vec{W}(1,0) = [1,1] \tag{16}$$

or, from 12

$$\vec{W}(1,0) = \vec{e}_r+\vec{e}_\theta \tag{17}$$

In this case, the polar basis vectors $\vec{e}_r$ and $\vec{e}_\theta$ happen to be the same as $\vec{e}_x$ and $\vec{e}_y$, so $\vec{W}$ has the same components in both systems.

Finally, suppose we try $r=2$, $\theta=\frac{\pi}{3}$. Then

$$\vec{W}\left(2,\frac{\pi}{3}\right) = \left[\frac{1}{2}\left(1+\sqrt{3}\right),\frac{1}{4}\left(1-\sqrt{3}\right)\right] \tag{18}$$

The basis vectors here are

$$\begin{aligned}\vec{e}_r &= \frac{1}{2}\vec{e}_x+\frac{\sqrt{3}}{2}\vec{e}_y\\ \vec{e}_\theta &= -\sqrt{3}\vec{e}_x+\vec{e}_y\end{aligned} \tag{19}$$

So we have (using Maple to do the algebra)

$$\begin{aligned}\vec{W}\left(2,\frac{\pi}{3}\right) &= \frac{1}{2}\left(1+\sqrt{3}\right)\left(\frac{1}{2}\vec{e}_x+\frac{\sqrt{3}}{2}\vec{e}_y\right)+\\ &\quad\frac{1}{4}\left(1-\sqrt{3}\right)\left(-\sqrt{3}\vec{e}_x+\vec{e}_y\right) \qquad (20)\\ &= \vec{e}_x+\vec{e}_y \qquad (21)\end{aligned}$$

In fact, combining 10 and 12 we see that in general (again using Maple to do the trig simplification)

$$\vec{W}(r,\theta) = (\cos\theta + \sin\theta)(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) + \frac{1}{r}(-\sin\theta + \cos\theta)(-r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y) \tag{22}$$

$$= \vec{e}_x + \vec{e}_y \tag{23}$$

Thus no matter what location in the polar plane we choose, the vector $\vec{W}$ is always the same as the original $[1,1]$ in rectangular coordinates.

Example 2. We can do the same exercise with a more complicated vector such as

$$\vec{V}_{\text{rect}} = \left[x^2 + 3y, y^2 + 3x\right] \tag{24}$$

We first write $\vec{V}$ in polar coordinates using 2:

$$\vec{V}_{\text{rect}} = \left[r^2\cos^2\theta + 3r\sin\theta, r^2\sin^2\theta + 3r\cos\theta\right] \tag{25}$$

We now apply 8 and 7 to get

$$\vec{V}_{\text{polar}} = \left[r^2\left(\cos^3\theta + \sin^3\theta\right) + 6r\sin\theta\cos\theta, r\left(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta\right) + 3\left(\cos^2\theta - \sin^2\theta\right)\right] \tag{26}$$

If we multiply out 26 together with 12 (using Maple, as it's a bit of a mess), we do indeed get 25 back again.

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