VERTICAL PARTICLE MOTION

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And now for a general relativistic look at the standard physics problem of throwing an object up in a gravitational field. Suppose we start at radius $r = r_0$ and throw up the object so that it comes to rest momentarily at $r = r_1$ before turning around and falling back to r_0. What is the total proper time (as measured by the object) in this trip? We're working in the Schwarzschild metric.

We begin by working out the energy e. For an object at rest at $r = r_1$:

$$e = \left(1 - \frac{2GM}{r_1}\right)\frac{dt}{d\tau} \tag{1}$$

$$= \left(1 - \frac{2GM}{r_1}\right)u^t \tag{2}$$

$$= \left(1 - \frac{2GM}{r_1}\right)\left(1 - \frac{2GM}{r_1}\right)^{-1/2} \tag{3}$$

$$= \left(1 - \frac{2GM}{r_1}\right)^{1/2} \tag{4}$$

The radial equation of motion in the Schwarzschild metric is

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{\ell^2}{r^2} - GM\left(\frac{1}{r} + \frac{\ell^2}{r^3}\right) = \frac{1}{2}\left(e^2 - 1\right) \tag{5}$$

For radial motion, $\ell = 0$ so we get

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(e^2 - 1\right) + \frac{GM}{r} \tag{6}$$

$$= GM\left(\frac{1}{r} - \frac{1}{r_1}\right) \tag{7}$$

$$= GM\frac{r_1 - r}{rr_1} \tag{8}$$

To find the time that elapses on the upward leg of the journey, we must evaluate

$$\Delta\tau=\sqrt{\frac{r_1}{2GM}}\int_{r_0}^{r_1}\sqrt{\frac{r}{r_1-r}}dr \tag{9}$$

This is a bit of a nasty integral. Using Maple, we get

$$\Delta\tau=\sqrt{\frac{r_1}{2GM}}\left[\frac{r_1}{2}\left(\frac{\pi}{2}-\arcsin\left(\frac{2r_0}{r_1}-1\right)\right)+\sqrt{r_0\left(r_1-r_0\right)}\right] \tag{10}$$

A couple of quick checks on this result. First, since r_1, r_0 and GM (in relativistic units) all have the dimensions of length (or time), the units check out. Second, if $r_1=r_0$, the object doesn't move at all since the start and end points are the same, and we have

$$\Delta\tau=\sqrt{\frac{r_1}{2GM}}\left[\frac{r_1}{2}\left(\frac{\pi}{2}-\arcsin\left(1\right)\right)+0\right]=0 \tag{11}$$

since $\arcsin(1)=\frac{\pi}{2}$. So this checks out as well.

As r_1 increases from r_0, the argument $\frac{2r_0}{r_1}-1$ of the arcsin in 10 decreases from 1 (at $r_1=r_0$) to -1 (as $r_1\to\infty$), so the arcsin starts at $\frac{\pi}{2}$ and moves to $-\frac{\pi}{2}$ as $r_1\to\infty$.

We can write 10 as

$$\Delta\tau=\frac{r_0^{3/2}}{u^{3/2}\sqrt{2GM}}\left[\frac{1}{2}\left(\frac{\pi}{2}-\arcsin\left(2u-1\right)\right)+\sqrt{u\left(1-u\right)}\right] \tag{12}$$

where

$$u\equiv\frac{r_0}{r_1} \tag{13}$$

The total time for going up and then down again is twice this, so we have

$$2\Delta\tau=\frac{r_0^{3/2}}{u^{3/2}\sqrt{2GM}}\left[\left(\frac{\pi}{2}-\arcsin\left(2u-1\right)\right)+2\sqrt{u\left(1-u\right)}\right] \tag{14}$$

Fig. 1 shows $\frac{\sqrt{2GM}}{r_0^{3/2}}\Delta\tau$ as a function of u as u varies from 1 (where the particle doesn't move at all) down to around 0.2, where $r_1=5r_0$.

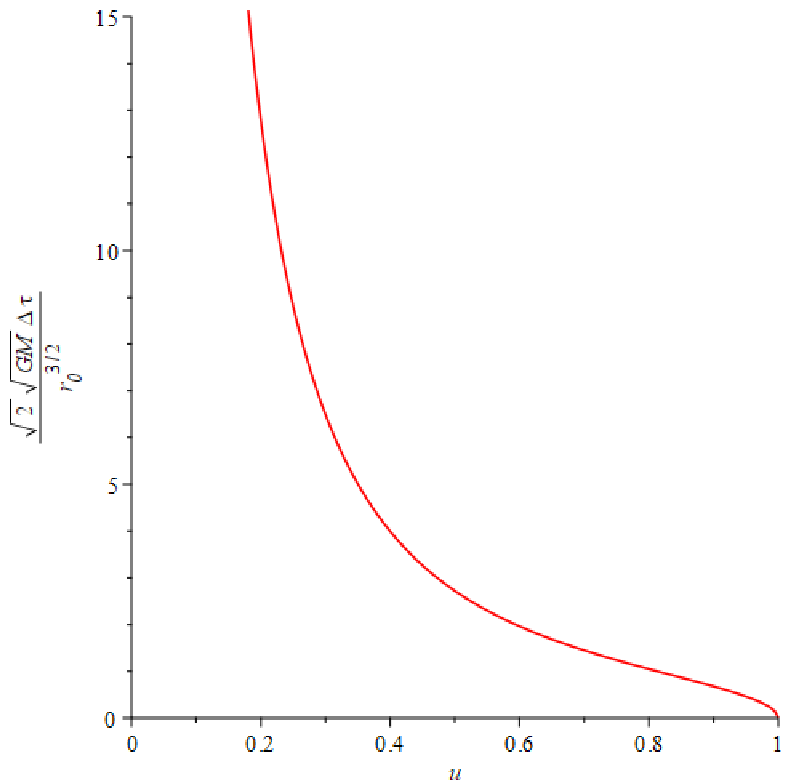

FIGURE 1. Plot of $\frac{\sqrt{2GM}}{r_0^{3/2}}\Delta\tau$ as a function of u for u varying between 1 and ≈ 0.2.