

## VOLUME ELEMENT IN TERMS OF METRIC DETERMINANT

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Zee shows with his eqn 23 in Chapter I.5 of his *Einstein Gravity* book that the volume element in  $D$ -dimensional space is invariant if multiplied by the square root of the determinant of the metric matrix. That is, the quantity

$$d^D x \sqrt{g} \quad (1)$$

is invariant, where

$$g = \det g_{\mu\nu} \quad (2)$$

An example of this is in two dimensions using polar coordinates, where the metric is

$$ds^2 = dr^2 + r^2 d\theta^2 \quad (3)$$

The metric is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \quad (4)$$

with determinant

$$g = r^2 \quad (5)$$

The 'volume' element here is an element of area, and is given by

$$d^2 x \sqrt{g} = r dr d\theta \quad (6)$$

For the unit circle, an element of length is obtained from 3 with  $dr = 0$ , so, with  $r = 1$

$$ds = r d\theta = d\theta \quad (7)$$

Technically, a line element on the unit circle is a one-dimensional space with  $ds^2 = d\theta^2$ , and  $g_{\mu\nu}$  has only one element, namely  $g_{\theta\theta} = 1$ . Thus the determinant is  $g = 1$  and the formula 1 holds with  $D = 1$ .

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Now in 3 dimensions, we consider the metric

$$ds^2 = a^2 dx^2 + b^2 dy^2 + c^2 dz^2 \quad (8)$$

so that

$$g_{\mu\nu} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} \quad (9)$$

$$g = \det g_{\mu\nu} = (abc)^2 \quad (10)$$

In this case, a volume element consists of

$$(a dx)(b dy)(c dz) = abc dx dy dz = \sqrt{g} d^3x \quad (11)$$