WORMHOLE METRIC

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As an example of embedding a 2-d surface in 3-d flat space we present another analysis of the inverse cosh example given earlier. The general metric is given as

$$ds^2 = -dt^2 + dr^2 + \left(b^2 + r^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

where b is a constant. For $b = 0$, this reduces to the ordinary spherical metric.

Here, we'll examine the 'slice' of this metric at constant t and $\theta = \frac{\pi}{2} =$ constant. The metric thus becomes

$$d\Sigma^2 = dr^2 + \left(b^2 + r^2\right) d\phi^2 \tag{2}$$

As before, this has cylindrical symmetry, since $d\Sigma^2$ is not changed if we increase or decrease the azimuthal angle ϕ. We therefore want to embed this 2-d surface in 3-d flat space with the cylindrical metric

$$dS^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2 \tag{3}$$

where ρ is the perpendicular distance from the z axis, ψ is the azimuthal angle and z is the distance above (or below, if it's negative) the plane $z = 0$.

We need to relate these coordinates to the quantities r and ϕ in 2. The azimuthal angle is the same in both cases so we have

$$\psi = \phi \tag{4}$$

$$d\psi = d\phi \tag{5}$$

Because of the axial symmetry, $z = z(r)$ and $\rho = \rho(r)$. Calculating differentials

$$\begin{aligned} d\rho^2 &= \left(\frac{d\rho}{dr}\right)^2 dr^2 \\ dz^2 &= \left(\frac{dz}{dr}\right)^2 dr^2 \end{aligned} \tag{6}$$

so we have from 3

$$dS^2 = \left[\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + \rho^2 d\phi^2 \tag{7}$$

Comparing with 2 we have the two equations

$$\begin{gathered}\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 = 1 \\ \rho^2 = b^2 + r^2\end{gathered} \tag{8}$$

We can use the second of these equations to eliminate ρ in favour of r, and we have

$$\rho = \sqrt{r^2 + b^2} \tag{9}$$

$$\frac{d\rho}{dr} = \frac{r}{\sqrt{r^2 + b^2}} \tag{10}$$

We therefore have a single equation for z:

$$\frac{dz}{dr} = \pm\sqrt{1 - \frac{r^2}{r^2 + b^2}} \tag{11}$$

Integrating we get (using Maple)

$$z = \pm b \ln\left(r + \sqrt{r^2 + b^2}\right) \tag{12}$$

In order to plot the surface, however, we need z as a function of ρ, since ρ is the cylindrical radial coordinate. From 8 we have

$$r = \sqrt{\rho^2 - b^2} \tag{13}$$

so

$$z(\rho) = \pm b \ln\left(\sqrt{\rho^2 - b^2} + \rho\right) \tag{14}$$

In order for the logarithm to be real, we must have $\rho \geq b$. Plotting both signs of z gives the surface in Fig. 1.

It appears that it represents a surface connected by a 'throat' which would be interpreted as a wormhole between two asymptotically flat spaces on either side. If we view the surface from above (Fig. 2) we see that the throat is a hole between the two surfaces on either side.

Unfortunately, no known physical space has this metric, so we can't use it to design an actual wormhole.

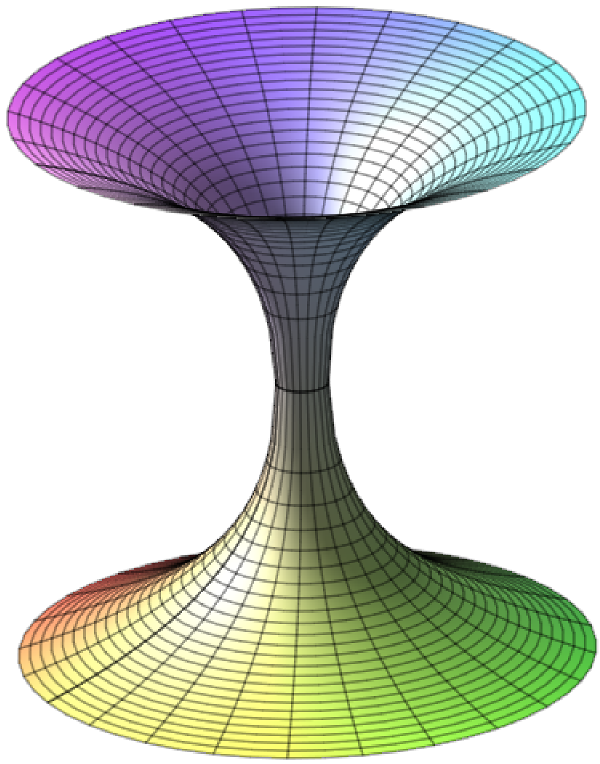

FIGURE 1. Wormhole surface. The upper blue-purple surface is $z=\ln\left(\sqrt{\rho^{2}-1}+\rho\right)$ and the lower yellow-green surface is $z=-\ln\left(\sqrt{\rho^{2}-1}+\rho\right)$.

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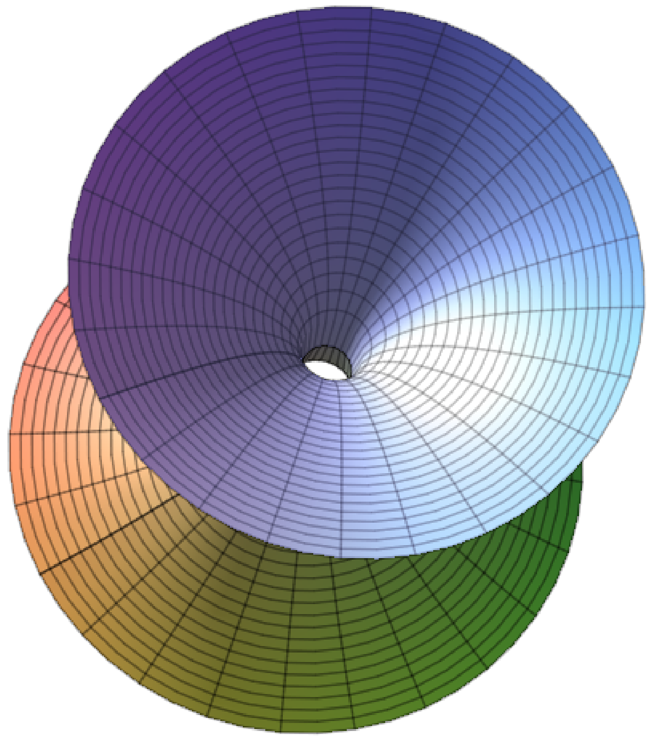

FIGURE 2. Wormhole surface from Fig. 1, looking down from above.