NEGATIVE HEAT CAPACITY IN GRAVITATIONAL SYSTEMS; ESTIMATING THE SUN'S TEMPERATURE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 1.55.

Suppose we have two identical masses m in a circular orbit of radius r about their centre of mass. By equating centripetal and gravitational forces, we have

$$\frac{mv^2}{r}=\frac{Gm^2}{4r^2} \tag{0.1}$$

The total kinetic and potential energies of the system are

$$K = 2\times\frac{mv^2}{2}=mv^2 \tag{0.2}$$

$$V = -\frac{Gm^2}{2r} \tag{0.3}$$

Therefore from 0.1

$$K=\frac{Gm^2}{4r}=-\frac{V}{2} \tag{0.4}$$

This is a special case of the virial theorem for gravitational orbits, which gives a relation between the average potential and kinetic energies:

$$\langle V\rangle=-2\langle K\rangle \tag{0.5}$$

The total energy of a gravitational system is therefore

$$U=\langle K\rangle+\langle V\rangle=-\langle K\rangle \tag{0.6}$$

The total energy is negative, which indicates that it is gravitationally bound and won't fly apart with time. However, this has a curious consequence in that, if we increase the energy of the system by an amount such that U is still negative, the kinetic energy must actually *decrease*.

For a system such as a star that is bound by gravitational forces and contains many particles, we can apply the equipartition theorem. Because of the high temperature within a star, the atomic nuclei become dissociated from their electrons, so the only degrees of freedom available to each particle are the translational degrees of freedom, meaning that the average kinetic energy of each particle is $\frac{3}{2}kT$. Therefore, for a system with N particles

$$U = -K = -\frac{3}{2}NkT \tag{0.7}$$

This means that the heat capacity is actually negative:

$$C = \frac{\partial U}{\partial T} = -\frac{3}{2}Nk \tag{0.8}$$

[Note that for a star, $C = C_V = C_P$ since even if the star's volume changes as energy is added to it, no expansion work is done since the star expands into a vacuum where the pressure is zero. However, gravitational work is done, so I'm not sure how that will affect the formula.]

At this point, Schroeder asks us to use dimensional analysis to get a formula for the potential energy of a star. I'm not quite sure what he means, but we can get an estimate of the potential energy as follows.

Suppose the star has a uniform density ρ and a radius R. Then the potential energy of a thin shell of the star at radius r is determined by the portion of the star inside this radius (the parts of the star outside r have no net force on anything inside; the proof of this is similar to that in electrostatics using Gauss's law since gravity is also an inverse square force). The portion of the star inside the radius r acts as a point mass M_r at the centre of the star. That is

$$\begin{aligned} dV &= -\frac{GM_r\,dm}{r} && (0.9)\\ &= -\frac{G}{r}\left(\frac{4}{3}\pi r^3\rho\right)\left(4\pi r^2\rho\,dr\right) && (0.10)\\ &= -\frac{16\pi^2 G}{3}\rho^2 r^4 dr && (0.11) \end{aligned}$$

We can integrate this to get V:

$$V = -\frac{16\pi^2 G}{3}\rho^2 \int_0^R r^4 dr \tag{0.12}$$

$$= -\frac{16\pi^2 G}{15}\rho^2 R^5 \tag{0.13}$$

$$= -\frac{3G}{5R}\left(\frac{4}{3}\pi R^3 \rho\right)^2 \tag{0.14}$$

$$= -\frac{3GM^2}{5R} \tag{0.15}$$

The average kinetic energy is therefore

$$\langle K\rangle = -\frac{1}{2}V = \frac{3GM^2}{10R} = \frac{3}{2}NkT \tag{0.16}$$

We can use this to get an estimate of the temperature of the Sun, whose mass is 2×10^{30} kg and radius is $R = 7\times10^{8}$ m. Taking the Sun to be composed of equal numbers of bare protons and electrons (and neglecting the mass of the electron compared that of the proton), we can estimate N:

$$N = \frac{2M}{m_p} \tag{0.17}$$

The temperature estimate is

$$T = \frac{GM^2}{5RNk} \tag{0.18}$$

$$= \frac{GMm_p}{10Rk} \tag{0.19}$$

$$= \frac{\left(6.67\times10^{-11}\right)\left(2\times10^{30}\right)\left(1.67\times10^{-27}\right)}{10\left(7\times10^{8}\right)\left(1.38\times10^{-23}\right)} \tag{0.20}$$

$$= 2.31\times10^{6}\text{ K} \tag{0.21}$$

This is within currently accepted values for the interior of the sun. The core of the Sun is estimated to be around 15×10^{6} K, but drops to between 2×10^{6} K and 7×10^{6} K as we get beyond half the Sun's radius from the core.