

## HEAT EQUATION

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 1.62.

The concepts of thermal conductivity and specific heat capacity can be combined to derive the heat equation, which governs how heat spreads through an object with a non-uniform temperature distribution. We'll derive the one-dimensional version of the heat equation.

Suppose we have a bar of material with some initial temperature distribution  $T(x, 0)$ , where  $T$  is a function of position  $x$  along the bar and time  $t$ , so  $T(x, 0)$  is the initial state of the bar at  $t = 0$ . Consider two adjacent slices in the bar, each of width  $\Delta x$ . The first slice is bounded by  $x_1$  and  $x_2$  and the second slice by  $x_2$  and  $x_3$ . According to heat conduction equation, the amount of heat  $Q_2$  flowing into the second slice from the first slice in time interval  $\Delta t$  is

$$(0.1) \quad \frac{Q_2}{\Delta t} = -k_t A \frac{T_2 - T_1}{\Delta x}$$

where  $k_t$  is the thermal conductivity and  $A$  is the cross-sectional area of the bar.

Similarly, the amount of heat flowing out of the second slice on the other side is

$$(0.2) \quad \frac{Q_1}{\Delta t} = -k_t A \frac{T_3 - T_2}{\Delta x}$$

The difference  $Q_2 - Q_1$  is stored in the second slice and will cause a change  $\Delta T$  in temperature within the slice. If the heat capacity of the material is  $c$  and its mass density is  $\rho$  then

$$(0.3) \quad \frac{Q_2 - Q_1}{(A\Delta x) c\rho} = \Delta T$$

Plugging in the values for  $Q_1$  and  $Q_2$  we get

$$(0.4) \quad \frac{\Delta T}{\Delta t} = \frac{k_t}{c\rho} \left[ \frac{T_1 - 2T_2 + T_3}{(\Delta x)^2} \right]$$

In the limit  $\Delta x \rightarrow 0$ , the quantity in brackets goes to  $\partial^2 T / \partial x^2$ .

If you haven't seen this form before, the argument goes like this. For some function  $f(x)$ , the second derivative is defined as

$$(0.5) \quad \frac{d^2 f}{dx^2} \equiv \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}$$

$$(0.6) \quad = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \frac{f(x + 2\Delta x) - f(x + \Delta x) - [f(x + \Delta x) - f(x)]}{\Delta x}$$

$$(0.7) \quad = \lim_{\Delta x \rightarrow 0} \frac{f(x + 2\Delta x) - 2f(x + \Delta x) + f(x)}{(\Delta x)^2}$$

This has the same form as 0.4. Thus in the limit, we get the heat equation

$$(0.8) \quad \frac{\partial T}{\partial t} = \frac{k_t}{c\rho} \frac{\partial^2 T}{\partial x^2}$$

One solution of this equation is

$$(0.9) \quad T(x, t) = T_0 + \frac{A}{\sqrt{t}} e^{-x^2/4Kt}$$

where

$$(0.10) \quad K \equiv \frac{k_t}{c\rho}$$

This can be verified by taking the derivatives, and we find that

$$(0.11) \quad \frac{\partial^2 T}{\partial x^2} = \frac{A(x^2 - 2Kt)}{4K^2 t^{5/2}} e^{-x^2/4Kt}$$

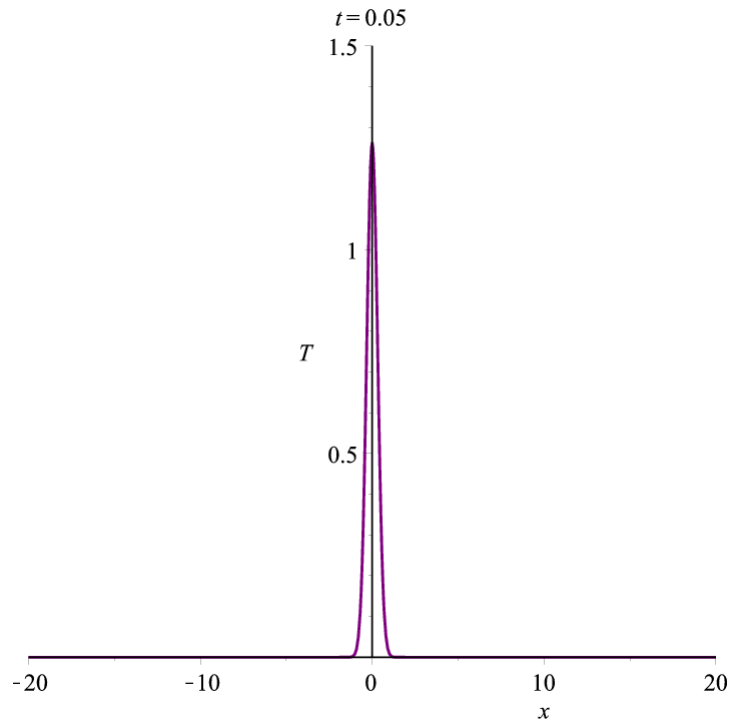
$$(0.12) \quad \frac{\partial T}{\partial t} = \frac{A(x^2 - 2Kt)}{4Kt^{5/2}} e^{-x^2/4Kt} = K \frac{\partial^2 T}{\partial x^2}$$

The function 0.9 with  $T_0 = 0$  and  $A = 1/2\sqrt{\pi K}$  is actually a delta function in the limit  $t \rightarrow 0$ . Using Maple, we find that

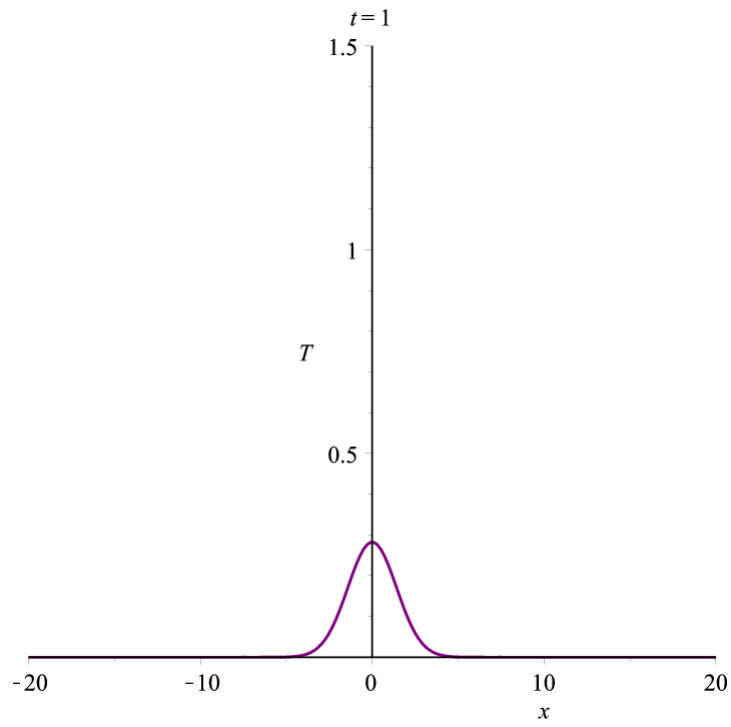
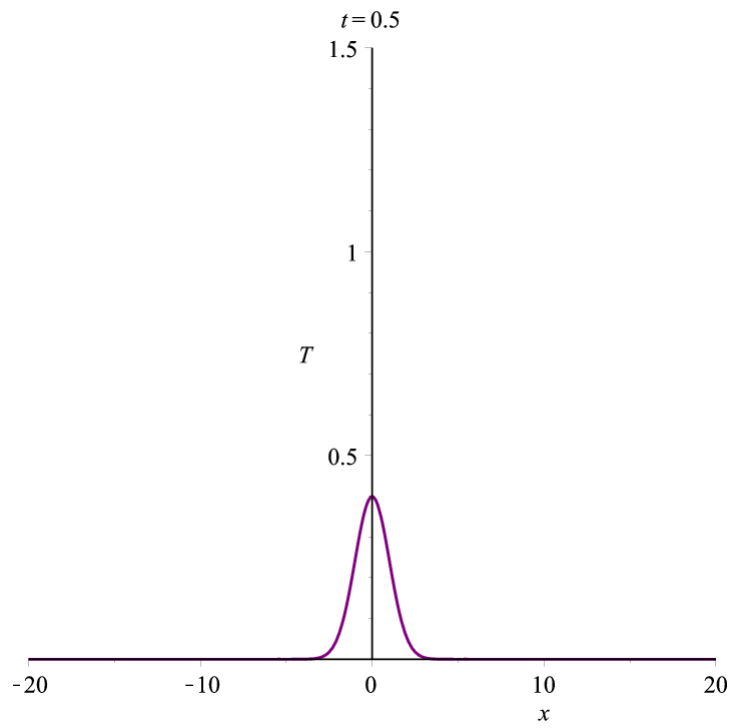
$$(0.13) \quad \frac{1}{2\sqrt{\pi K}} \int_{-\infty}^{\infty} \frac{e^{-x^2/4Kt}}{\sqrt{t}} dx = 1$$

For any  $x \neq 0$ ,  $\lim_{t \rightarrow 0} \frac{e^{-x^2/4Kt}}{\sqrt{t}} = 0$  so in this limit, the function has an infinitely high spike at  $x = 0$  and an integral of 1, which are the conditions of a delta function. As time increases, the function becomes a standard Gaussian curve which gradually spreads out until as  $t \rightarrow \infty$ , it becomes

zero, so  $\lim_{t \rightarrow \infty} T(x, t) = T_0$ . This is what we'd expect since the heat will gradually diffuse over the bar until everywhere is at the same background temperature. Here are some plots of  $T(x, t)$  for various values of  $t$ , with  $A = 1/2\sqrt{\pi K}$ ,  $K = 1$  and  $T_0 = 0$ :



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