HEAT EQUATION

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 1.62.

The concepts of thermal conductivity and specific heat capacity can be combined to derive the heat equation, which governs how heat spreads through an object with a non-uniform temperature distribution. We'll derive the one-dimensional version of the heat equation.

Suppose we have a bar of material with some initial temperature distribution $T(x,0)$, where T is a function of position x along the bar and time t, so $T(x,0)$ is the initial state of the bar at $t=0$. Consider two adjacent slices in the bar, each of width Δx. The first slice is bounded by x_1 and x_2 and the second slice by x_2 and x_3. According to heat conduction equation, the amount of heat Q_2 flowing into the second slice from the first slice in time interval Δt is

$$\frac{Q_2}{\Delta t}=-k_tA\frac{T_2-T_1}{\Delta x} \tag{1}$$

where k_t is the thermal conductivity and A is the cross-sectional area of the bar.

Similarly, the amount of heat flowing out of the second slice on the other side is

$$\frac{Q_1}{\Delta t}=-k_tA\frac{T_3-T_2}{\Delta x} \tag{2}$$

The difference Q_2-Q_1 is stored in the second slice and will cause a change ΔT in temperature within the slice. If the heat capacity of the material is c and its mass density is ρ then

$$\frac{Q_2-Q_1}{(A\Delta x)\,c\rho}=\Delta T \tag{3}$$

Plugging in the values for Q_1 and Q_2 we get

$$\frac{\Delta T}{\Delta t}=\frac{k_t}{c\rho}\left[\frac{T_1-2T_2+T_3}{(\Delta x)^2}\right] \tag{4}$$

In the limit $\Delta x \to 0$, the quantity in brackets goes to $\partial^2 T/\partial x^2$.

If you haven't seen this form before, the argument goes like this. For some function $f(x)$, the second derivative is defined as

$$\frac{d^2 f}{dx^2} \equiv \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x} \tag{5}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{f(x+2\Delta x) - f(x+\Delta x) - [f(x+\Delta x) - f(x)]}{\Delta x} \tag{6}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+2\Delta x) - 2f(x+\Delta x) + f(x)}{(\Delta x)^2} \tag{7}$$

This has the same form as 4. Thus in the limit, we get the heat equation

$$\frac{\partial T}{\partial t} = \frac{k_t}{c\rho} \frac{\partial^2 T}{\partial x^2} \tag{8}$$

One solution of this equation is

$$T(x,t) = T_0 + \frac{A}{\sqrt{t}} e^{-x^2/4Kt} \tag{9}$$

where

$$K \equiv \frac{k_t}{c\rho} \tag{10}$$

This can be verified by taking the derivatives, and we find that

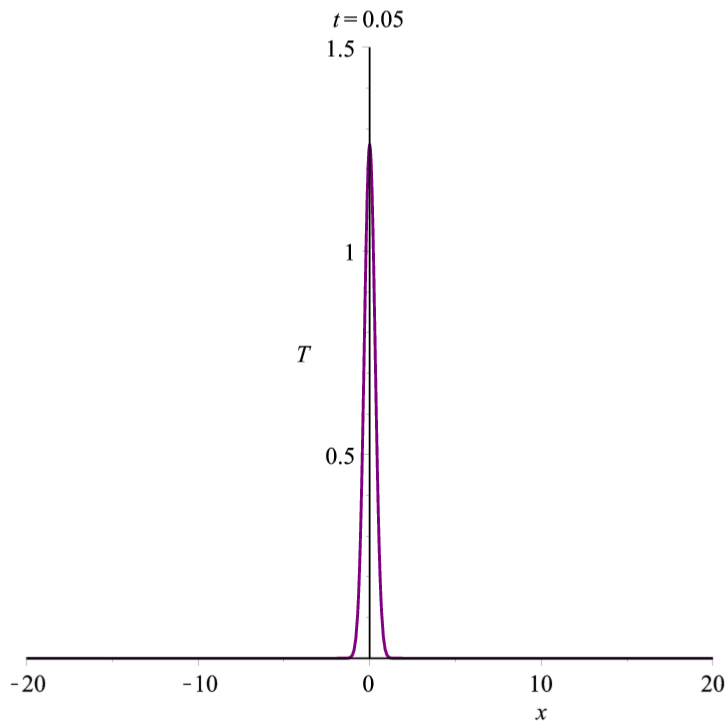
$$\frac{\partial^2 T}{\partial x^2} = \frac{A\left(x^2 - 2Kt\right)}{4K^2 t^{5/2}} e^{-x^2/4Kt} \tag{11}$$

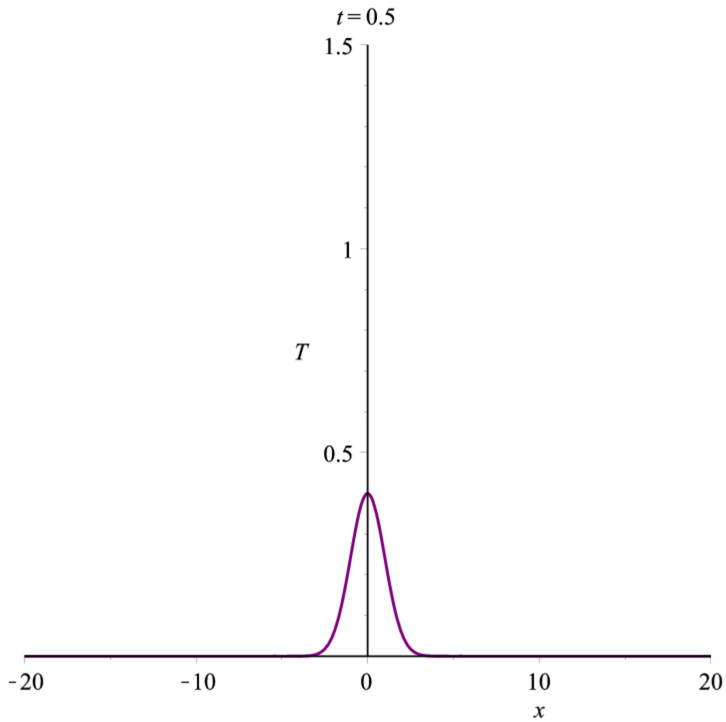
$$\frac{\partial T}{\partial t} = \frac{A\left(x^2 - 2Kt\right)}{4Kt^{5/2}} e^{-x^2/4Kt} = K \frac{\partial^2 T}{\partial x^2} \tag{12}$$

The function 9 with $T_0 = 0$ and $A = 1/2\sqrt{\pi K}$ is actually a delta function in the limit $t \to 0$. Using Maple, we find that

$$\frac{1}{2\sqrt{\pi K}} \int_{-\infty}^{\infty} \frac{e^{-x^2/4Kt}}{\sqrt{t}} dx = 1 \tag{13}$$

For any $x \neq 0$, $\lim_{t\to 0} \frac{e^{-x^2/4Kt}}{\sqrt{t}} = 0$ so in this limit, the function has an infinitely high spike at $x = 0$ and an integral of 1, which are the conditions of a delta function. As time increases, the function becomes a standard Gaussian curve which gradually spreads out until as $t \to \infty$, it becomes zero, so $\lim_{t\to\infty} T(x,t) = T_0$. This is what we'd expect since the heat will gradually diffuse over the bar until everywhere is at the same background temperature. Here are some plots of $T(x,t)$ for various values of t, with $A = 1/2\sqrt{\pi K}$, $K = 1$ and $T_0 = 0$:

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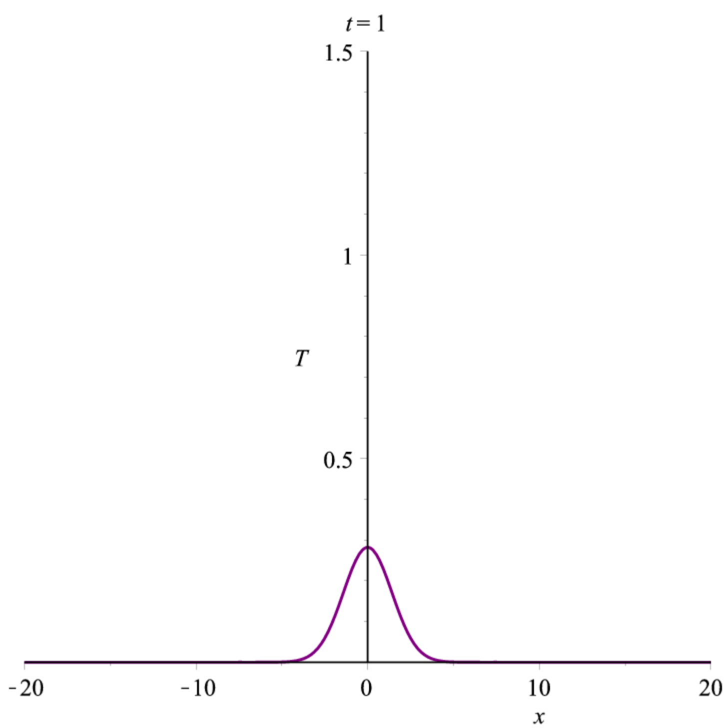
1.5
1
T
0.5
-20
-10
0
10
20
x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1.5
1
T
0.5
-20
-10
0
10
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x

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