INTERACTING EINSTEIN SOLIDS

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 2.8.

We've seen how to count micro- and macrostates in an Einstein solid. In a solid containing N oscillators and q quanta of energy, there are $\binom{q+N-1}{q}$ possible microstates. Consider now what happens if we have two such solids, A and B, containing N_A and N_B oscillators and q_A and q_B quanta of energy. Each solid has its own set of microstates, but suppose that the solids can exchange energy quanta on a timescale that is quite long compared with the times over which quanta can travel between oscillators within each solid. We're assuming that total energy q_A+q_B is conserved in this process, but that q_A and q_B each can vary within this constraint (that is, the solids can exchange energy quanta between them). We'd like to investigate the probabilities of the various divisions of energy between the two solids.

For any particular partition of the quanta, that is, for particular values of q_A and q_B, the total number of microstates available to the compound system is

$$\Omega_{total}=\Omega_A\Omega_B=\binom{q_A+N_A-1}{q_A}\binom{q_B+N_B-1}{q_B} \tag{1}$$

This is true because for each microstate in solid A, we could have any of the Ω_B microstates in system B.

If we consider all the possible partitions of quanta, the total number of microstates available to the compound system is the sum of this quantity over all possible values of q_A (remember $q_B=q_{total}-q_A$ so q_B isn't an independent variable since q_{total} is a constant). Looked at another way, we can view the compound solid as a single solid with N_A+N_B oscillators and q_A+q_B quanta, so the overall number of microstates is

$$\Omega_{overall}=\binom{q_A+q_B+N_A+N_B-1}{q_A+q_B} \tag{2}$$

The fundamental assumption of statistical mechanics is that if we look at the system at any instant of time, we are equally likely to find it in any one of these $\Omega_{overall}$ microstates. The question then becomes: given the division of the solid into two systems with the number of oscillators N_A and N_B in each solid fixed, what is the most likely distribution of the energy quanta between the two solids? That is, what is the most likely value of q_A?

For relatively small systems, we can calculate these probabilities by brute force by just working out the binomial coefficients. For larger systems (ones containing a number of particles typical of macroscopic objects), this is no longer feasible so we need to resort to approximations. But for now, we can work out an example with manageable numbers.

Before we begin, we need one final bit of terminology. We'll refer to the *macrostate* of a compound solid as a particular division of the quanta between the two solids, without regard to how the quanta within each solid are distributed among the oscillators in that solid. In other words, each possible value of q_A defines one macrostate. Since the possible values of q_A are $0, 1, \ldots, q_A + q_B$, there are $q_A + q_B + 1$ possible macrostates in such a system.

Example. Suppose $N_A = N_B = 10$ and $q_A + q_B = 20$. There are therefore 21 possible macrostates. The number of microstates is

$$\Omega_{overall} = \binom{20+20-1}{20} = 6.89 \times 10^{10} \tag{3}$$

The probability that all the energy is in solid A is

$$\frac{1}{\Omega_{overall}} \binom{20+10-1}{20} \binom{0+10-1}{0} = \frac{10^7}{6.89 \times 10^{10}} = 1.45 \times 10^{-4} \tag{4}$$

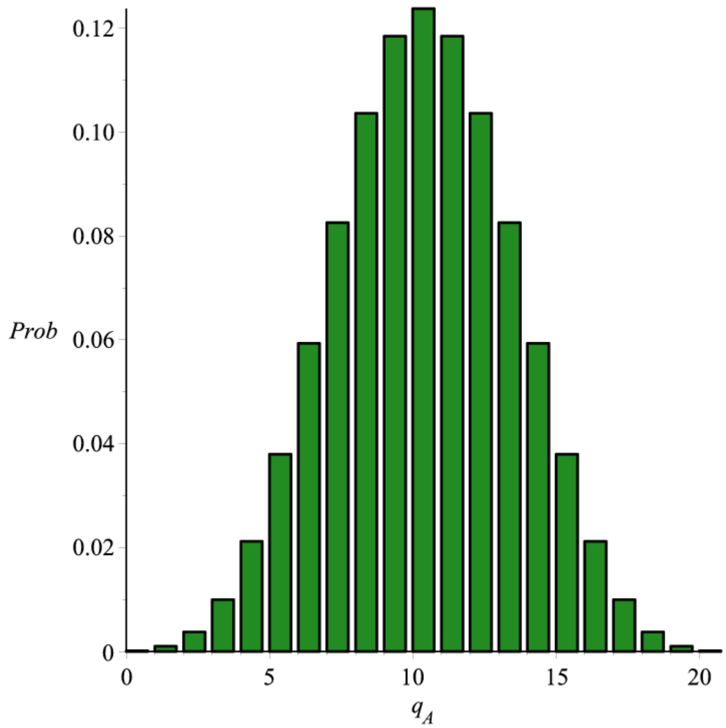
The probability that $q_A = q_B = 10$ (that is, the energy is evenly distributed) is

$$\frac{1}{\Omega_{overall}} \binom{10+10-1}{10} \binom{10+10-1}{10} = 0.1238 \tag{5}$$

The probability for a general value of q_A is

$$Prob = \frac{1}{\Omega_{overall}} \binom{10+q_A-1}{q_A} \binom{10+20-q_A-1}{20-q_A} \tag{6}$$

A bar chart of the probabilities is



It's much more likely that the quanta will distribute themselves equally between the two solids, and once such a state is achieved, it's unlikely that it will return to a state where one solid has a lot more quanta than the other. That is, a state where the quanta are distributed equally is said to be *irreversible*.

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