

EINSTEIN SOLIDS: MULTIPLICITY OF LARGE SYSTEMS

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problems 2.17-2.19.

The number of microstates in an Einstein solid with N oscillators and q energy quanta is

$$\Omega = \binom{q+N-1}{q} \quad (1)$$

For any macroscopic solid, both q and N are large numbers (on the order of Avogadro's number, or 10^{23}) so the factorials in Ω are very large numbers, not calculable on most computers. To get estimates of Ω we can use Stirling's approximation for the factorials. The derivation of this approximation for the high temperature case $q \gg N$ (lots more energy quanta than oscillators) is given in Schroeder's book, so I'll deal here with the low temperature case $q \ll N$.

Writing out the binomial coefficient

$$\binom{q+N-1}{q} = \frac{(q+N-1)!}{q!(N-1)!} \quad (2)$$

$$\approx \frac{(q+N)!}{q!N!} \quad (3)$$

where we've the fact that if we multiply a very large number like $(q+N-1)!$ by a merely large number like N , the original very large number is essentially unchanged.

We can now take logs and use Stirling's approximation for the log of a factorial

$$\ln n! \approx n \ln n - n \quad (4)$$

We get

$$\ln \Omega \approx (q+N) \ln (q+N) - q - N - q \ln q + q - N \ln N + N \quad (5)$$

$$= (q+N) \ln (q+N) - q \ln q - N \ln N \quad (6)$$

If we now make the assumption that $q \ll N$, we get

$$\ln \Omega \approx (q+N) \ln \left[N \left(1 + \frac{q}{N} \right) \right] - q \ln q - N \ln N \quad (7)$$

$$= (q+N) \left[\ln N + \ln \left(1 + \frac{q}{N} \right) \right] - q \ln q - N \ln N \quad (8)$$

$$\approx q \ln N + (q+N) \frac{q}{N} - q \ln q \quad (9)$$

$$= q \ln \frac{N}{q} + q + \frac{q^2}{N} \quad (10)$$

$$\approx q \ln \frac{N}{q} + q \quad (11)$$

where to get the third line we've used the approximation $\ln(1+x) \approx x$ for $|x| \ll 1$, and in the last line we've neglected the q^2/N term in the $q \ll N$ limit. Exponentiating this result gives the approximate value for Ω :

$$\Omega \approx \left(\frac{Ne}{q} \right)^q \quad (12)$$

[The corresponding result in the high temperature case is $\Omega \approx (qe/N)^N$ which could have been predicted easily, since q and N appear symmetrically in the approximation 3.]

This result applies also to the two-state paramagnet with N magnetic dipoles and N_\downarrow energy quanta, since the system is formally equivalent to an Einstein solid (we're distributing the energy quanta among dipoles rather than oscillators). The multiplicity of the paramagnet is then

$$\Omega \approx \left(\frac{Ne}{N_\downarrow} \right)^{N_\downarrow} \quad (13)$$

Finally, we can use Stirling's approximation on 2 directly to get an approximation for the case where N and q are any large numbers, without one necessarily being much larger than the other. We have

$$\Omega = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!(N-1)!} \quad (14)$$

$$= \frac{1}{q!} \frac{N}{N!} \frac{(q+N)!}{(q+N)} \quad (15)$$

$$= \frac{N}{q+N} \frac{(q+N)!}{q!N!} \quad (16)$$

Stirling's approximation for a large factorial is

$$n! \approx \sqrt{2\pi n} n^n e^{-n} \quad (17)$$

so we get

$$\Omega \approx \frac{N}{q+N} \frac{\sqrt{2\pi(q+N)} (q+N)^{q+N} e^{-(q+N)}}{2\pi\sqrt{qN} q^q N^N e^{-(q+N)}} \quad (18)$$

$$= \sqrt{\frac{N}{2\pi q(q+N)}} \frac{(q+N)^{q+N}}{q^q N^N} \quad (19)$$

$$= \sqrt{\frac{N}{2\pi q(q+N)}} \left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N \quad (20)$$

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