

EINSTEIN SOLIDS: MULTIPLICITY OF LARGE SYSTEMS

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problems 2.17-2.19.

The number of microstates in an Einstein solid with N oscillators and q energy quanta is

$$(0.1) \quad \Omega = \binom{q+N-1}{q}$$

For any macroscopic solid, both q and N are large numbers (on the order of Avogadro's number, or 10^{23}) so the factorials in Ω are very large numbers, not calculable on most computers. To get estimates of Ω we can use Stirling's approximation for the factorials. The derivation of this approximation for the high temperature case $q \gg N$ (lots more energy quanta than oscillators) is given in Schroeder's book, so I'll deal here with the low temperature case $q \ll N$.

Writing out the binomial coefficient

$$(0.2) \quad \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!(N-1)!}$$

$$(0.3) \quad \approx \frac{(q+N)!}{q!N!}$$

where we've the fact that if we multiply a very large number like $(q+N-1)!$ by a merely large number like N , the original very large number is essentially unchanged.

We can now take logs and use Stirling's approximation for the log of a factorial

$$(0.4) \quad \ln n! \approx n \ln n - n$$

We get

$$(0.5) \quad \ln \Omega \approx (q+N) \ln(q+N) - q - N - q \ln q + q - N \ln N + N$$

$$(0.6) \quad = (q+N) \ln(q+N) - q \ln q - N \ln N$$

If we now make the assumption that $q \ll N$, we get

$$(0.7) \quad \ln \Omega \approx (q+N) \ln \left[N \left(1 + \frac{q}{N} \right) \right] - q \ln q - N \ln N$$

$$(0.8) \quad = (q+N) \left[\ln N + \ln \left(1 + \frac{q}{N} \right) \right] - q \ln q - N \ln N$$

$$(0.9) \quad \approx q \ln N + (q+N) \frac{q}{N} - q \ln q$$

$$(0.10) \quad = q \ln \frac{N}{q} + q + \frac{q^2}{N}$$

$$(0.11) \quad \approx q \ln \frac{N}{q} + q$$

where to get the third line we've used the approximation $\ln(1+x) \approx x$ for $|x| \ll 1$, and in the last line we've neglected the q^2/N term in the $q \ll N$ limit. Exponentiating this result gives the approximate value for Ω :

$$(0.12) \quad \Omega \approx \left(\frac{Ne}{q} \right)^q$$

[The corresponding result in the high temperature case is $\Omega \approx (qe/N)^N$ which could have been predicted easily, since q and N appear symmetrically in the approximation 0.3.]

This result applies also to the two-state paramagnet with N magnetic dipoles and N_\downarrow energy quanta, since the system is formally equivalent to an Einstein solid (we're distributing the energy quanta among dipoles rather than oscillators). The multiplicity of the paramagnet is then

$$(0.13) \quad \Omega \approx \left(\frac{Ne}{N_\downarrow} \right)^{N_\downarrow}$$

Finally, we can use Stirling's approximation on 0.2 directly to get an approximation for the case where N and q are any large numbers, without one necessarily being much larger than the other. We have

$$(0.14) \quad \Omega = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!(N-1)!}$$

$$(0.15) \quad = \frac{1}{q!} \frac{N}{N!} \frac{(q+N)!}{(q+N)}$$

$$(0.16) \quad = \frac{N}{q+N} \frac{(q+N)!}{q!N!}$$

Stirling's approximation for a large factorial is

$$(0.17) \quad n! \approx \sqrt{2\pi n} n^n e^{-n}$$

so we get

$$(0.18) \quad \Omega \approx \frac{N}{q+N} \frac{\sqrt{2\pi(q+N)} (q+N)^{q+N} e^{-(q+N)}}{2\pi\sqrt{qN} q^q N^N e^{-(q+N)}}$$

$$(0.19) \quad = \sqrt{\frac{N}{2\pi q(q+N)}} \frac{(q+N)^{q+N}}{q^q N^N}$$

$$(0.20) \quad = \sqrt{\frac{N}{2\pi q(q+N)}} \left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N$$

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