EINSTEIN SOLIDS: MULTIPLICITY OF LARGE SYSTEMS

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problems 2.17-2.19.

The number of microstates in an Einstein solid with N oscillators and q energy quanta is

$$\Omega=\binom{q+N-1}{q} \tag{1}$$

For any macroscopic solid, both q and N are large numbers (on the order of Avogadro's number, or 10^{23}) so the factorials in Ω are very large numbers, not calculable on most computers. To get estimates of Ω we can use Stirling's approximation for the factorials. The derivation of this approximation for the high temperature case $q\gg N$ (lots more energy quanta than oscillators) is given in Schroeder's book, so I'll deal here with the low temperature case $q\ll N$.

Writing out the binomial coefficient

$$\begin{aligned}\binom{q+N-1}{q} &= \frac{(q+N-1)!}{q!\,(N-1)!} && (2)\\ &\approx \frac{(q+N)!}{q!N!} && (3)\end{aligned}$$

where we've the fact that if we multiply a very large number like $(q+N-1)!$ by a merely large number like N, the original very large number is essentially unchanged.

We can now take logs and use Stirling's approximation for the log of a factorial

$$\ln n!\approx n\ln n-n \tag{4}$$

We get

$$\begin{aligned}\ln\Omega &\approx (q+N)\ln(q+N)-q-N-q\ln q+q-N\ln N+N && (5)\\ &= (q+N)\ln(q+N)-q\ln q-N\ln N && (6)\end{aligned}$$

If we now make the assumption that $q\ll N$, we get

$$\begin{aligned}
\ln\Omega &\approx (q+N)\ln\left[N\left(1+\frac{q}{N}\right)\right]-q\ln q-N\ln N && (7)\\
&= (q+N)\left[\ln N+\ln\left(1+\frac{q}{N}\right)\right]-q\ln q-N\ln N && (8)\\
&\approx q\ln N+(q+N)\frac{q}{N}-q\ln q && (9)\\
&= q\ln\frac{N}{q}+q+\frac{q^2}{N} && (10)\\
&\approx q\ln\frac{N}{q}+q && (11)
\end{aligned}$$

where to get the third line we've used the approximation $\ln(1+x)\approx x$ for $|x|\ll 1$, and in the last line we've negelected the q^2/N term in the $q\ll N$ limit. Exponentiating this result gives the approximate value for Ω:

$$\Omega\approx\left(\frac{Ne}{q}\right)^q \tag{12}$$

[The corresponding result in the high temperature case is $\Omega\approx(qe/N)^N$ which could have been predicted easily, since q and N appear symmetrically in the approximation 3.]

This result applies also to the two-state paramagnet with N magnetic dipoles and $N_\downarrow$ energy quanta, since the system is formally equivalent to an Einstein solid (we're distributing the energy quanta among dipoles rather than oscillators). The multiplicity of the paramagnet is then

$$\Omega\approx\left(\frac{Ne}{N_\downarrow}\right)^{N_\downarrow} \tag{13}$$

Finally, we can use Stirling's approximation on 2 directly to get an approximation for the case where N and q are any large numbers, without one necessarily being much larger than the other. We have

$$\begin{aligned}
\Omega=\binom{q+N-1}{q} &= \frac{(q+N-1)!}{q!\,(N-1)!} && (14)\\
&= \frac{1}{q!}\frac{N}{N!}\frac{(q+N)!}{(q+N)} && (15)\\
&= \frac{N}{q+N}\frac{(q+N)!}{q!N!} && (16)
\end{aligned}$$

Stirling's approximation for a large factorial is

$$n! \approx \sqrt{2\pi n} n^n e^{-n} \tag{17}$$

so we get

$$\begin{aligned}
\Omega &\approx \frac{N}{q+N}\frac{\sqrt{2\pi(q+N)}(q+N)^{q+N}e^{-(q+N)}}{2\pi\sqrt{qN}q^q N^N e^{-(q+N)}} && (18)\\
&= \sqrt{\frac{N}{2\pi q(q+N)}}\frac{(q+N)^{q+N}}{q^q N^N} && (19)\\
&= \sqrt{\frac{N}{2\pi q(q+N)}}\left(\frac{q+N}{q}\right)^q\left(\frac{q+N}{N}\right)^N && (20)
\end{aligned}$$

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