INTERACTING EINSTEIN SOLIDS: SHARPNESS OF THE MULTIPLICITY FUNCTION

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problems 2.20 - 2.21.

The number of microstates in an Einstein solid with N oscillators and q energy quanta is

$$\Omega=\binom{q+N-1}{q} \tag{0.1}$$

For large systems at high temperatures, so that $q\gg N$ we have the approximate formula

$$\Omega\approx\left(\frac{qe}{N}\right)^{N} \tag{0.2}$$

If we now have two such solids and allow them to interact, the number of microstates for the combined system for any given macrostate (that is, a given division of the total energy $q=q_A+q_B$ between the two solids) is just the product of the numbers for the two separate solids:

$$\Omega=\left(\frac{q_Ae}{N_A}\right)^{N_A}\left(\frac{q_Be}{N_B}\right)^{N_B} \tag{0.3}$$

We've already seen that the most probable state for a pair of interacting solids is the state in which the energy quanta are distributed evenly between the two systems, so that $q_A/q_B=N_A/N_B$. Our goal in this post is to investigate how likely it is that the distribution of energy will deviate significantly from this most probable state.

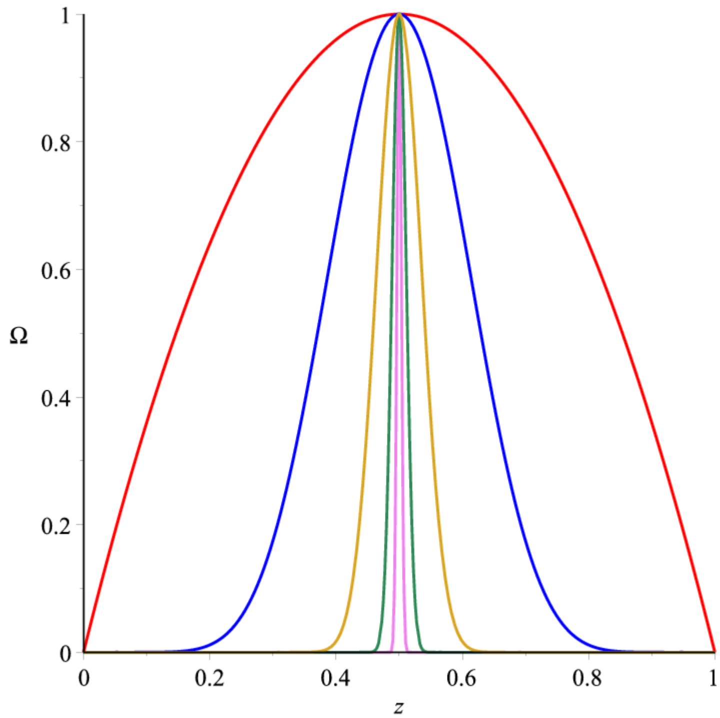
To make things simpler, we'll take $N_A=N_B=N$ so that both solids are the same size. Then

$$\Omega=\left(\frac{e}{N}\right)^{2N}\left(q_Aq_B\right)^{N}=\left(\frac{e}{N}\right)^{2N}\left(q_A\left(q-q_A\right)\right)^{N} \tag{0.4}$$

With N held constant, the shape of this curve is determined by the $\left(q_A\left(q-q_A\right)\right)^{N}$ factor. If we pull out a factor of q^2, we get

$$(q_A(q-q_A))^N = q^{2N}\left(\frac{q_A}{q}\left(1-\frac{q_A}{q}\right)\right)^N \equiv q^{2N}(z(1-z))^N \tag{0.5}$$

We can get a feel for how the curve's shape changes as we increase N by plotting $(z(1-z))^N$ for several values of N. z itself ranges from 0 to 1 and $z(1-z)$ has a maximum value of 0.25 at $z=0.5$, so we can scale the graph to a vertical range of 0 to 1 by inserting a factor of 4 inside the parentheses. That is, we plot $(4z(1-z))^N$, as shown:



The curves are for $N=1$ (red), $N=10$ (blue), $N=100$ (yellow), $N=1000$ (green) and $N=10000$ (violet).

The actual width of the curve can be approximated by the calculation given by Schroeder in section 2.4. The peak of the curve occurs at $q_A=\frac{q}{2}$ so we can investigate the behaviour for $q_A=\frac{q}{2}+x$ and $q_B=\frac{q}{2}-x$ where $x \ll q$. Schroeder shows that Ω becomes, near the peak:

$$\Omega \approx \left(\frac{eq}{2N}\right)^{2N} e^{-N(2x/q)^2} \tag{0.6}$$

which is a Gaussian curve. The width can be defined as the distance in x between the points where the curve is $\frac{1}{e}$ of its maximum value, which occurs when

$$N\left(\frac{2x}{q}\right)^2 = 1 \tag{0.7}$$

$$x = \frac{q}{2\sqrt{N}} \tag{0.8}$$

so the width of the curve is twice this, or $q/\sqrt{N}$. Since we're assuming $q \gg N$, this width is still a large number, but since the total width of the graph is $q+1$, the width of the peak as a fraction of the total width of the graph is around $1/\sqrt{N}$, which for any macroscopic value of N, is vanishingly small. For example, if N is on the order of Avogadro's number, 10^{23}, and we drew the graph so that the total width of the graph fits on a page 10 cm wide, the width of the central peak is around 3×10^{-13} m, which is about $\frac{1}{100}$ the size of a hydrogen atom.

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