INTERACTING EINSTEIN SOLIDS: RECTANGULAR PEAK IN MULTIPLICITY GRAPH

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 2.22.

In a system composed of two interacting Einstein solids, the multiplicity function, which gives the number of microstates as a function of the number of energy quanta q_A in solid A, is very sharply peaked about the point where the quanta are evenly distributed between the two solids. The width of the peak is approximately $q/\sqrt{N}$ for a system containing a total of q energy quanta and N oscillators.

We can get another, somewhat rougher, estimate of this width by first calculating the total number of microstates Ω_{total} accessible to the system (that is, the total number of microstates summed over all possible macrostates), and then finding the number of microstates Ω_{mp} in the most probable macrostate (even distribution of energy quanta). If we then assume that the peak in the graph is rectangular rather than Gaussian, then the width w of the peak is found from the area of the rectangular peak, according to

$$w=\frac{\Omega_{total}}{\Omega_{mp}} \tag{1}$$

To apply this, let's consider a simple system where the two solids each have N oscillators and the total number of quanta is $q=2N$. For large N and q, we can use the approximation derived earlier for the number of microstates in a solid with q quanta and n oscillators (I'm using a lowercase n here to distinguish it from the N in the problem):

$$\Omega\approx\sqrt{\frac{n}{2\pi q\left(q+n\right)}}\left(\frac{q+n}{q}\right)^{q}\left(\frac{q+n}{n}\right)^{n} \tag{2}$$

To find Ω_{total}, we can combine the two solids into one composite solid since we're interested in *all* microstates, no matter how the quanta are divided up between the two solids. In this case $q=2N$ and $n=2N$, so we get

$$\Omega_{total} \approx \sqrt{\frac{2N}{2\pi(2N)(4N)}}\left(\frac{4N}{2N}\right)^{2N}\left(\frac{4N}{2N}\right)^{2N} \tag{3}$$

$$= \frac{2^{4N}}{\sqrt{8\pi N}} \tag{4}$$

To find Ω_{mp}, we need to separate the solid into its constituent parts A and B, and assign quanta so that $q_A = q_B = \frac{q}{2} = N$. The total number of microstates for this particular macrostate is then

$$\Omega_{mp} = \left[\sqrt{\frac{N}{2\pi(N)(N)}}\left(\frac{2N}{N}\right)^{N}\left(\frac{2N}{N}\right)^{N}\right]^2 \tag{5}$$

$$= \frac{2^{4N}}{4\pi N} \tag{6}$$

The term in square brackets is Ω for one of the solids, since it contains $n = N$ oscillators and $q_A = N$ energy quanta. Since the other solid is identical, we just square the result to get Ω_{mp}.

The width of the peak is then

$$w = \frac{4\pi N}{\sqrt{8\pi N}} = \sqrt{2\pi N} \tag{7}$$

[The Gaussian width is $q/\sqrt{N} = 2\sqrt{N}$ and since $\sqrt{2\pi N} \approx 2.5\sqrt{N}$ the rectangular approximation isn't actually all that bad.]

The total number of macrostates in the two-solid system is $q + 1 = 2N + 1 \approx 2N$, so the fraction of macrostates that have reasonably large probabilities is

$$\frac{\sqrt{2\pi N}}{2N} = \sqrt{\frac{\pi}{2N}} \tag{8}$$

For a macroscopic solid with $N = 10^{23}$, this fraction comes out to around 4×10^{-12}. This shows just how unlikely it is that a macroscopic system will ever be found with an energy distribution significantly different from the most probable case.

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