

## PARAMAGNETS & COIN FLIPS: PEAK AND WIDTH OF MULTIPLICITY FUNCTION

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problems 2.23 - 2.24.

For a couple of examples of Stirling's approximation we'll return to the two-state paramagnet. First, suppose we have a paramagnet with  $N = 10^{23}$  dipoles and that exactly half are in the spin-up state, so that  $N_{\uparrow} = \frac{1}{2} \times 10^{23}$ . The number of microstates in this system is

$$\begin{aligned}
 (1) \quad \Omega &= \frac{N!}{(N/2)!(N/2)!} \\
 (2) \quad &\approx \frac{\sqrt{N} N^N e^{-N}}{\sqrt{2\pi} (N/2) (N/2)^N e^{-N}} \\
 (3) \quad &= \frac{2^{N+1}}{\sqrt{2\pi N}} \\
 (4) \quad &\approx 2^N
 \end{aligned}$$

where the last line is true for  $N$  values around  $10^{23}$  since this makes  $2^N$  a *very* large number, so that dividing by the square root won't change things much. [Note that in this approximation,  $\Omega$  for the peak is actually equal to the total number of microstates summed over *all* macrostates, so the result says that an equal distribution is virtually certain.]

If the system was in a different microstate a billion times per second over a period of ten billion years, it would explore

$$(5) \quad 10^9 \times 10^{10} \times 365.25 \times 24 \times 3600 = 3.16 \times 10^{26}$$

microstates. The number of available microstates is

$$(6) \quad 2^{10^{23}} \approx 10^{3 \times 10^{22}}$$

so the actual number of microstates visited in 10 billion years is a vanishingly small proportion of those available. In theory, if we wait long enough, I suppose it's *possible* to visit all the "accessible" microstates, but we'd have

to wait an almost infinite time (at least on a time scale of the current age of the universe) for this to happen.

In Schroeder's problem 2.24, he asks us to work out the width of the multiplicity peak for a two-state paramagnet. However, I'm not sure quite why this question makes sense, since I would imagine that the applied magnetic field would determine the total energy of the paramagnet, and thus the actual macrostate the paramagnet is in. Thus talking about the width of multiplicity peak doesn't seem to make sense here. However, as part of the question asks about flipping coins, we'll go with that interpretation.

If we flip  $N$  coins, the multiplicity function peaks sharply at a number  $N_h$  of heads equal to  $N/2$ . The actual multiplicity function in this case has the value given in 1, so this is the height of the peak.

To get an estimate of the width of the peak, we can define

$$(7) \quad x \equiv N_h - \frac{N}{2}$$

to be the deviation of the observed number of heads from the most probable value of  $\frac{N}{2}$ . The multiplicity function in terms of  $x$  is therefore

$$(8) \quad \Omega(x) = \frac{N!}{\left(\frac{N}{2} - x\right)! \left(\frac{N}{2} + x\right)!}$$

$$(9) \quad \approx \frac{\sqrt{N} N^N}{\sqrt{2\pi} \sqrt{\frac{N}{2} - x} \sqrt{\frac{N}{2} + x} \left(\frac{N}{2} - x\right)^{\frac{N}{2} - x} \left(\frac{N}{2} + x\right)^{\frac{N}{2} + x}}$$

$$(10) \quad = \frac{\sqrt{N} N^N}{\sqrt{2\pi} \sqrt{\left(\frac{N}{2}\right)^2 - x^2} \left(\left(\frac{N}{2}\right)^2 - x^2\right)^{\frac{N}{2}} \left(\frac{N}{2} - x\right)^{-x} \left(\frac{N}{2} + x\right)^x}$$

We can now work with logarithms and consider the case of large  $N$ , so we can neglect the square root terms. Thus the approximation becomes

(11)

$$\Omega(x) \approx \frac{N^N}{\left(\left(\frac{N}{2}\right)^2 - x^2\right)^{\frac{N}{2}} \left(\frac{N}{2} - x\right)^{-x} \left(\frac{N}{2} + x\right)^x}$$

(12)

$$\ln \Omega = N \ln N - \frac{N}{2} \ln \left( \left( \frac{N}{2} \right)^2 - x^2 \right) - x \ln \frac{\frac{N}{2} + x}{\frac{N}{2} - x}$$

(13)

$$= N \ln N - \frac{N}{2} \ln \left[ \frac{N^2}{4} \left( 1 - \frac{4x^2}{N^2} \right) \right] - x \ln \frac{1 + \frac{2x}{N}}{1 - \frac{2x}{N}}$$

(14)

$$= N \ln N - \frac{N}{2} \ln \left( \frac{N}{2} \right)^2 - \frac{N}{2} \ln \left( 1 - \frac{4x^2}{N^2} \right) - x \left[ \ln \left( 1 + \frac{2x}{N} \right) - \ln \left( 1 - \frac{2x}{N} \right) \right]$$

(15)

$$= \ln 2^N - \frac{N}{2} \ln \left( 1 - \frac{4x^2}{N^2} \right) - x \left[ \ln \left( 1 + \frac{2x}{N} \right) - \ln \left( 1 - \frac{2x}{N} \right) \right]$$

We can now use the approximation  $\ln(1+z) \approx z$  for  $|z| \ll 1$  and get

$$(16) \quad \ln \Omega \approx \ln 2^N + \frac{N}{2} \frac{4x^2}{N^2} - x \left( \frac{2x}{N} - \left( -\frac{2x}{N} \right) \right)$$

$$(17) \quad = \ln 2^N - \frac{2x^2}{N}$$

Thus the multiplicity function near the peak is approximately

$$(18) \quad \Omega(x) \approx 2^N e^{-2x^2/N}$$

Note that for  $x = 0$ , we get the peak height in 4. [If we'd retained the square root factors in 10, we'd get the  $\sqrt{2\pi N}$  factor back.]

Thus the peak falls to  $1/e$  of its peak value when

$$(19) \quad x = \sqrt{\frac{N}{2}}$$

so the width of the peak is twice this, or

$$(20) \quad w = \sqrt{2N}$$

If we flipped  $10^6$  coins, a result of 501,000 heads is just outside the peak, since  $x = \sqrt{\frac{10^6}{2}} \approx 700$ , so this wouldn't be a particularly surprising result. However, a result of 510,000 heads is well outside the peak and *would* be a surprising result.

## PINGBACKS

Pingback: Random walks and diffusion

Pingback: Multiplicity of a 2-dim ideal gas

Pingback: Entropy

Pingback: Two-state paramagnet: numerical solution

Pingback: Two-state paramagnet: analytic solution