MULTIPLICITY OF A 2-DIM IDEAL GAS

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 2.26.

Having looked at counting the number of microstates in systems like coin flipping and the Einstein solid, we can now look at counting the microstates in an ideal gas.

At first glance, this might seem to be impossible, since in classical physics at least, a gas molecule confined within a volume V with a fixed energy U can be in an infinite number of states, since it could be at any location within the volume and the components of its momentum could have any values subject to the constraint that the kinetic energy of the molecule is $p^2/2m$. That's true, and to be able to count the number of states of a gas molecule, we need to use quantum mechanics. Because of the uncertainty principle, the location and momentum of a molecule can be determined only up to regions Δx (in position space) and Δp (in momentum space) such that $\Delta x \Delta p \geq \hbar/2$. Schroeder's derivation of the number of microstates available to an ideal gas begins with this principle, or at least with the approximation

$$\Delta x \Delta p \approx h \tag{1}$$

Because the gas is confined within a volume V, the situation is somewhat similar to the particle in a box or infinite square well. A particle in such a potential well is restricted to discrete energy states, and also restricted to a location between the infinitely high barriers at either end. It might seem that for a state with fixed energy, the momentum would be precisely known, but in fact, because the particle can move either to the right or left, the momentum doesn't have a fixed value, and a particle in such a fixed energy state does in fact satisfy the uncertainty principle.

Schroeder's argument is that, in one dimension, a molecule can be localized to a particular region Δx in position space and a particular momentum interval Δp in momentum space, so if the particle is confined to a location range L and momentum range L_p, the number of available states in position space is $L/\Delta x$ and in momentum space is $L_p/\Delta p$, and, since the momentum and position ranges are independent, the total number of microstates is

$$\Omega = \frac{LL_p}{\Delta x \Delta p} = \frac{LL_p}{h} \tag{2}$$

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Schroeder then develops the theory for a 3-d ideal gas in detail so we won't go through that again here. Rather, we'll look at an analogous 2-dimensional case. In 2-d, we can imagine the molecule confined to an area A in position space and another area A_p in momentum space. Since the energy of the molecule is fixed, the momentum space is constrained by the condition

$$p_x^2 + p_y^2 = 2mU \tag{3}$$

which is the equation of a circle. Thus the 'area' A_p is actually the circumference of the circle:

$$A_p = 2\pi\sqrt{2mU} \tag{4}$$

Since there is an uncertainty of h for the products of position and momentum in each of the two dimensions, we get

$$\Omega = \frac{A}{h^2} 2\pi\sqrt{2mU} \tag{5}$$

To generalize this to the case with N gas molecules, we note that in position space, the locations of all the molecules are independent, so we'll get a factor of A^N/h^{2N}. The momenta are constrained by the condition

$$\sum_{i=1}^{N} \left(p_{i_x}^2 + p_{i_y}^2 \right) = 2mU \tag{6}$$

where the sum index i extends over the N molecules, so the momenta are *not* independent. The total 'volume' of momentum space is actually the 'area' of a $2N$ dimensional hypersphere, which Schroeder derives in his Appendix B and gives as

$$\text{area} = \frac{2\pi^{d/2}}{(\frac{d}{2} - 1)!} r^{d-1} \tag{7}$$

where r is the radius and d is the dimension. Thus we get, using $d = 2N$ and $r = \sqrt{2mU}$:

$$\Omega = \frac{A^N}{h^{2N}} \frac{2\pi^N}{(N-1)!} \left(\sqrt{2mU} \right)^{2N-1} \tag{8}$$

This would be the formula if the gas molecules were distinguishable. However, one of the principles of quantum mechanics is that elementary particles of the same type are actually identical. In that case, with N molecules, interchanging any pair of them will leave the microstate unchanged, so the above formula actually overcounts the number of states by

$N!$. The actual number of microstates for a 2-d ideal gas of indistinguishable particles is therefore

$$\Omega = \frac{A^N}{N! h^{2N}} \frac{2\pi^N}{(N-1)!} \left(\sqrt{2mU}\right)^{2N-1} \tag{9}$$

Since N is a large number (on the order of 10^{23}), we can approximate this by

$$\Omega \approx \frac{(\pi A)^N}{(N!)^2 h^{2N}} \left(\sqrt{2mU}\right)^{2N} \tag{10}$$

In SI units, h is a small number (around 10^{-34}) and N is a large number, so the number of microstates will typically be very large.

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