

## ENTROPY OF AN IDEAL GAS; SACKUR-TETRODE EQUATION

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problems 2.31 - 2.33.

The entropy of a substance is given as

$$(0.1) \quad S = k \ln \Omega$$

where  $\Omega$  is the number of microstates accessible to the substance.

For a 3-d ideal gas, this is given by Schroeder's equation 2.40:

$$(0.2) \quad \Omega \approx \frac{V^N (2\pi mU)^{3N/2}}{h^{3N} N! (3N/2)!}$$

where  $V$  is the volume,  $U$  is the energy,  $N$  is the number of molecules,  $m$  is the mass of a single molecule and  $h$  is Planck's constant. We can further approximate this formula by using Stirling's approximation for the factorials:

$$(0.3) \quad N! \approx \sqrt{2\pi N} N^N e^{-N}$$

$$(0.4) \quad (3N/2)! \approx \sqrt{3\pi N} \left(\frac{3N}{2}\right)^{3N/2} e^{-3N/2}$$

We get

$$(0.5) \quad \Omega \approx \frac{V^N (\pi mU)^{3N/2}}{h^{3N}} \frac{2^{3N} e^{5N/2}}{\sqrt{63}^{3N/2} \pi N^{5N/2+1}}$$

When  $N$  is large, we can throw away a couple of factors and take the logarithm:

$$(0.6) \quad \Omega \approx \frac{V^N (\pi m U)^{3N/2} 2^{3N} e^{5N/2}}{h^{3N} 3^{3N/2} N^{5N/2}}$$

$$(0.7) \quad \ln \Omega = N \ln \left( V \left( \frac{\pi m U}{3} \right)^{3/2} \left( \frac{2}{h} \right)^3 \frac{1}{N^{5/2}} \right) + \frac{5N}{2}$$

$$(0.8) \quad = N \left[ \ln \left( \frac{V}{N} \left( \frac{4\pi m U}{3Nh^2} \right)^{3/2} \right) + \frac{5}{2} \right]$$

This gives the entropy of an ideal gas as

$$(0.9) \quad S = Nk \left[ \ln \left( \frac{V}{N} \left( \frac{4\pi m U}{3Nh^2} \right)^{3/2} \right) + \frac{5}{2} \right]$$

which is known as the Sackur-Tetrode equation.

**Example 1.** A variant of this equation can be derived in a similar way for the 2-d ideal gas considered earlier. In that case we had

$$(0.10) \quad \Omega \approx \frac{(\pi A)^N}{(N!)^2 h^{2N}} (\sqrt{2mU})^{2N}$$

where  $A$  is the area of the gas. Using Stirling's approximation as before, we get

$$(0.11) \quad \Omega \approx \frac{(\pi A)^N}{2\pi N^{2N+1} e^{-2N} h^{2N}} (\sqrt{2mU})^{2N}$$

$$(0.12) \quad \approx \frac{(\pi A)^N}{N^{2N} e^{-2N} h^{2N}} (2mU)^N$$

$$(0.13) \quad S = k \ln \Omega = Nk \left[ \ln \frac{2\pi m A U}{(hN)^2} + 2 \right]$$

**Example 2.** Schroeder gives the entropy of a mole of helium at room temperature and atmospheric pressure as  $S = 126 \text{ J K}^{-1}$ . For another monatomic gas such as argon, we can work out the same thing. From the ideal gas law, at a pressure of  $1.01 \times 10^5 \text{ N m}^{-2}$  and temperature of 300 K, one mole occupies a volume of

$$(0.14) \quad V = \frac{nRT}{P} = \frac{(1)(8.31)(300)}{1.01 \times 10^5} = 0.025 \text{ m}^3$$

The internal energy of a monatomic gas is  $\frac{1}{2}kT$  per molecule per degree of freedom, so for one mole we have

$$(0.15) \quad U = \frac{3}{2}NkT = \frac{3}{2}nRT = 3739 \text{ J}$$

The mass of a mole of argon is  $39.948 \times 10^{-3}$  kg, so with  $N = 6.02 \times 10^{23}$  we have

$$(0.16) \quad m = \frac{39.948 \times 10^{-3}}{6.02 \times 10^{23}} = 6.64 \times 10^{-26} \text{ kg}$$

Plugging these values into 0.9, the entropy comes out to

$$(0.17) \quad S = (6.02 \times 10^{23}) (1.38 \times 10^{-23}) (\ln 1.02 \times 10^7 + 2.5)$$

$$(0.18) \quad = 155 \text{ J K}^{-1}$$

This is a bit higher than the value for helium because of argon's higher mass.

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