

ENERGY OF A SYSTEM WITH QUADRATIC DEGREES OF FREEDOM

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 3.6.

The definition of temperature in terms of entropy is

$$\frac{1}{T} \equiv \frac{\partial S}{\partial U} \quad (1)$$

Schroeder quotes a theorem that states that for any system with only quadratic degrees of freedom in the high temperature limit, the multiplicity Ω is proportional to $U^{Nf/2}$ where U is the energy, N is the number of molecules and f is the number of degrees of freedom per molecule. For an ideal gas, this is given by Schroeder's equation 2.40:

$$\Omega \approx \frac{V^N (2\pi mU)^{3N/2}}{h^{3N} N! (3N/2)!} \quad (2)$$

Since $f = 3$ for a monatomic ideal gas (3 translational degrees of freedom only), the theorem is valid here. Similarly, for an Einstein solid in the high temperature limit

$$\Omega \approx \left(\frac{qe}{N}\right)^N \quad (3)$$

and since q , the number of energy quanta, is proportional to U . Each oscillator in an Einstein solid is equivalent to a one-dimensional harmonic oscillator, which has two degrees of freedom. [Rather, it has two quadratic terms in its energy: $\frac{1}{2}kx^2$ for potential energy and $\frac{1}{2}mv^2$ for its kinetic energy. Each of these is interpreted as a 'degree of freedom'.] Thus again, $\Omega \propto U^{Nf/2} = U^N$.

In the general case, we have

$$\Omega = AU^{Nf/2} \quad (4)$$

for some constant A (that is, A doesn't depend on U). The entropy is therefore

$$S = k \ln \Omega = k \ln A + \frac{kNf}{2} \ln U \quad (5)$$

and the temperature is given by

$$\frac{1}{T} = \frac{kNf}{2U} \quad (6)$$

$$U = \frac{1}{2}NfkT \quad (7)$$

which is just what the equipartition theorem predicts ($\frac{1}{2}kT$ energy per degree of freedom).

The formula is valid only for large energies, since if U is small enough, Ω eventually becomes less than 1 and the entropy S can become negative, which isn't physically possible.