

## ENERGY OF A SYSTEM WITH QUADRATIC DEGREES OF FREEDOM

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 3.6.

The definition of temperature in terms of entropy is

$$(1) \quad \frac{1}{T} \equiv \frac{\partial S}{\partial U}$$

Schroeder quotes a theorem that states that for any system with only quadratic degrees of freedom in the high temperature limit, the multiplicity  $\Omega$  is proportional to  $U^{Nf/2}$  where  $U$  is the energy,  $N$  is the number of molecules and  $f$  is the number of degrees of freedom per molecule. For an ideal gas, this is given by Schroeder's equation 2.40:

$$(2) \quad \Omega \approx \frac{V^N (2\pi mU)^{3N/2}}{h^{3N} N! (3N/2)!}$$

Since  $f = 3$  for a monatomic ideal gas (3 translational degrees of freedom only), the theorem is valid here. Similarly, for an Einstein solid in the high temperature limit

$$(3) \quad \Omega \approx \left(\frac{qe}{N}\right)^N$$

and since  $q$ , the number of energy quanta, is proportional to  $U$ . Each oscillator in an Einstein solid is equivalent to a one-dimensional harmonic oscillator, which has two degrees of freedom. [Rather, it has two quadratic terms in its energy:  $\frac{1}{2}kx^2$  for potential energy and  $\frac{1}{2}mv^2$  for its kinetic energy. Each of these is interpreted as a 'degree of freedom'.] Thus again,  $\Omega \propto U^{Nf/2} = U^N$ .

In the general case, we have

$$(4) \quad \Omega = AU^{Nf/2}$$

for some constant  $A$  (that is,  $A$  doesn't depend on  $U$ ). The entropy is therefore

$$(5) \quad S = k \ln \Omega = k \ln A + \frac{kNf}{2} \ln U$$

and the temperature is given by

$$(6) \quad \frac{1}{T} = \frac{kNf}{2U}$$

$$(7) \quad U = \frac{1}{2}NfkT$$

which is just what the equipartition theorem predicts ( $\frac{1}{2}kT$  energy per degree of freedom).

The formula is valid only for large energies, since if  $U$  is small enough,  $\Omega$  eventually becomes less than 1 and the entropy  $S$  can become negative, which isn't physically possible.