TWO-STATE PARAMAGNET: ANALYTIC SOLUTION

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 3.19.

We've looked at a two-state paramagnet and generated some curves involving entropy, energy and temperature using numerical methods. Here we'll derive the analytic solution.

This is a system of N magnetic dipoles which, when placed in a magnetic field B, align themselves so that their magnetic moment μ points either parallel or antiparallel to the field. The energy of a dipole that is aligned with the field is lower, and we'll call it $-\mu B$, so that the antiparallel dipole has energy $+\mu B$, and the total energy of the system is

$$U = \mu B\left(N_{\downarrow}-N_{\uparrow}\right)=\mu B\left(N-2N_{\uparrow}\right) \tag{1}$$

$$N_{\uparrow} = \frac{1}{2}\left(N-\frac{U}{\mu B}\right) \tag{2}$$

where an up arrow indicates parallel alignment and a down arrow antiparallel.

The net magnetization is then

$$M=\mu\left(N_{\uparrow}-N_{\downarrow}\right)=-\frac{U}{B} \tag{3}$$

The multiplicity of states is the same as a set of N coins with $N_{\uparrow}$ heads, so

$$\Omega=\binom{N}{N_{\uparrow}}=\frac{N!}{N_{\uparrow}!\left(N-N_{\uparrow}\right)!} \tag{4}$$

We begin the analytic solution by looking at the entropy directly, and using Stirling's approximation.

$$\frac{S}{k}=\ln\Omega \tag{5}$$

$$\approx\left(N\ln N-N\right)-\left(N_{\uparrow}\ln N_{\uparrow}-N_{\uparrow}\right)-\left(\left(N-N_{\uparrow}\right)\ln\left(N-N_{\uparrow}\right)-\left(N-N_{\uparrow}\right)\right) \tag{6}$$

$$=N\ln N-N_{\uparrow}\ln N_{\uparrow}-\left(N-N_{\uparrow}\right)\ln\left(N-N_{\uparrow}\right) \tag{7}$$

We can now get the temperature from

$$\frac{1}{T} = \frac{\partial S}{\partial U} \tag{8}$$

Using the chain rule, 2 and 1 we get

$$\begin{aligned}
\frac{1}{T} &= \frac{\partial S}{\partial N_\uparrow}\frac{\partial N_\uparrow}{\partial U} && (9)\\
\frac{\partial S}{\partial N_\uparrow} &= k\left[-\ln N_\uparrow - 1 + \ln\left(N - N_\uparrow\right) + 1\right] && (10)\\
&= k\ln\frac{N - N_\uparrow}{N_\uparrow} && (11)\\
&= k\ln\frac{N + U/\mu B}{N - U/\mu B} && (12)\\
\frac{\partial N_\uparrow}{\partial U} &= -\frac{1}{2\mu B} && (13)\\
\frac{1}{T} &= -\frac{k}{2\mu B}\ln\frac{N + U/\mu B}{N - U/\mu B} && (14)\\
&= \frac{k}{2\mu B}\ln\frac{N - U/\mu B}{N + U/\mu B} && (15)
\end{aligned}$$

Exponentiating both sides gives

$$e^{2\mu B/kT} = \frac{N - U/\mu B}{N + U/\mu B} \tag{16}$$

which can be solved for U:

$$\begin{aligned}
\frac{U}{\mu B}\left[e^{2\mu B/kT} + 1\right] &= N\left[1 - e^{2\mu B/kT}\right] && (17)\\
U &= N\mu B\frac{1 - e^{2\mu B/kT}}{1 + e^{2\mu B/kT}} && (18)\\
&= N\mu B\frac{e^{-\mu B/kT} - e^{\mu B/kT}}{e^{-\mu B/kT} + e^{\mu B/kT}} && (19)\\
&= -N\mu B\tanh\frac{\mu B}{kT} && (20)
\end{aligned}$$

The magnetism is, from 3

$$M = N\mu\tanh\frac{\mu B}{kT} \tag{21}$$

Finally, the heat capacity at constant magnetic field is (using the derivative of the tanh function: $d\tanh x/dx = \text{sech}^2 x$):

$$C_B = \frac{\partial U}{\partial T} \tag{22}$$

$$= -N\mu B\left[\text{sech}^2\frac{\mu B}{kT}\right]\left[-\frac{\mu B}{kT^2}\right] \tag{23}$$

$$= \frac{Nk\left(\mu B/kT\right)^2}{\cosh^2\left(\mu B/kT\right)} \tag{24}$$

These analytic formulas give the curves we saw earlier using the numerical solution.

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