

TWO-STATE PARAMAGNET: ANALYTIC SOLUTION

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 3.19.

We've looked at a two-state paramagnet and generated some curves involving entropy, energy and temperature using numerical methods. Here we'll derive the analytic solution.

This is a system of N magnetic dipoles which, when placed in a magnetic field B , align themselves so that their magnetic moment μ points either parallel or antiparallel to the field. The energy of a dipole that is aligned with the field is lower, and we'll call it $-\mu B$, so that the antiparallel dipole has energy $+\mu B$, and the total energy of the system is

$$(0.1) \quad U = \mu B (N_{\downarrow} - N_{\uparrow}) = \mu B (N - 2N_{\uparrow})$$

$$(0.2) \quad N_{\uparrow} = \frac{1}{2} \left(N - \frac{U}{\mu B} \right)$$

where an up arrow indicates parallel alignment and a down arrow antiparallel.

The net magnetization is then

$$(0.3) \quad M = \mu (N_{\uparrow} - N_{\downarrow}) = -\frac{U}{B}$$

The multiplicity of states is the same as a set of N coins with N_{\uparrow} heads, so

$$(0.4) \quad \Omega = \binom{N}{N_{\uparrow}} = \frac{N!}{N_{\uparrow}! (N - N_{\uparrow})!}$$

We begin the analytic solution by looking at the entropy directly, and using Stirling's approximation.

(0.5)

$$\frac{S}{k} = \ln \Omega$$

(0.6)

$$\approx (N \ln N - N) - (N_{\uparrow} \ln N_{\uparrow} - N_{\uparrow}) - ((N - N_{\uparrow}) \ln (N - N_{\uparrow}) - (N - N_{\uparrow}))$$

(0.7)

$$= N \ln N - N_{\uparrow} \ln N_{\uparrow} - (N - N_{\uparrow}) \ln (N - N_{\uparrow})$$

We can now get the temperature from

$$(0.8) \quad \frac{1}{T} = \frac{\partial S}{\partial U}$$

Using the chain rule, 0.2 and 0.1 we get

$$(0.9) \quad \frac{1}{T} = \frac{\partial S}{\partial N_{\uparrow}} \frac{\partial N_{\uparrow}}{\partial U}$$

$$(0.10) \quad \frac{\partial S}{\partial N_{\uparrow}} = k [-\ln N_{\uparrow} - 1 + \ln (N - N_{\uparrow}) + 1]$$

$$(0.11) \quad = k \ln \frac{N - N_{\uparrow}}{N_{\uparrow}}$$

$$(0.12) \quad = k \ln \frac{N + U/\mu B}{N - U/\mu B}$$

$$(0.13) \quad \frac{\partial N_{\uparrow}}{\partial U} = -\frac{1}{2\mu B}$$

$$(0.14) \quad \frac{1}{T} = -\frac{k}{2\mu B} \ln \frac{N + U/\mu B}{N - U/\mu B}$$

$$(0.15) \quad = \frac{k}{2\mu B} \ln \frac{N - U/\mu B}{N + U/\mu B}$$

Exponentiating both sides gives

$$(0.16) \quad e^{2\mu B/kT} = \frac{N - U/\mu B}{N + U/\mu B}$$

which can be solved for U :

$$(0.17) \quad \frac{U}{\mu B} \left[e^{2\mu B/kT} + 1 \right] = N \left[1 - e^{2\mu B/kT} \right]$$

$$(0.18) \quad U = N\mu B \frac{1 - e^{2\mu B/kT}}{1 + e^{2\mu B/kT}}$$

$$(0.19) \quad = N\mu B \frac{e^{-\mu B/kT} - e^{\mu B/kT}}{e^{-\mu B/kT} + e^{\mu B/kT}}$$

$$(0.20) \quad = -N\mu B \tanh \frac{\mu B}{kT}$$

The magnetism is, from 0.3

$$(0.21) \quad M = N\mu \tanh \frac{\mu B}{kT}$$

Finally, the heat capacity at constant magnetic field is (using the derivative of the tanh function: $d \tanh x / dx = \text{sech}^2 x$):

$$(0.22) \quad C_B = \frac{\partial U}{\partial T}$$

$$(0.23) \quad = -N\mu B \left[\text{sech}^2 \frac{\mu B}{kT} \right] \left[-\frac{\mu B}{kT^2} \right]$$

$$(0.24) \quad = \frac{Nk(\mu B/kT)^2}{\cosh^2(\mu B/kT)}$$

These analytic formulas give the curves we saw earlier using the numerical solution.

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