

## TWO-STATE PARAMAGNET: ANALYTIC SOLUTION

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 3.19.

We've looked at a two-state paramagnet and generated some curves involving entropy, energy and temperature using numerical methods. Here we'll derive the analytic solution.

This is a system of  $N$  magnetic dipoles which, when placed in a magnetic field  $B$ , align themselves so that their magnetic moment  $\mu$  points either parallel or antiparallel to the field. The energy of a dipole that is aligned with the field is lower, and we'll call it  $-\mu B$ , so that the antiparallel dipole has energy  $+\mu B$ , and the total energy of the system is

$$U = \mu B (N_{\downarrow} - N_{\uparrow}) = \mu B (N - 2N_{\uparrow}) \quad (1)$$

$$N_{\uparrow} = \frac{1}{2} \left( N - \frac{U}{\mu B} \right) \quad (2)$$

where an up arrow indicates parallel alignment and a down arrow antiparallel.

The net magnetization is then

$$M = \mu (N_{\uparrow} - N_{\downarrow}) = -\frac{U}{B} \quad (3)$$

The multiplicity of states is the same as a set of  $N$  coins with  $N_{\uparrow}$  heads, so

$$\Omega = \binom{N}{N_{\uparrow}} = \frac{N!}{N_{\uparrow}! (N - N_{\uparrow})!} \quad (4)$$

We begin the analytic solution by looking at the entropy directly, and using Stirling's approximation.

$$\frac{S}{k} = \ln \Omega \quad (5)$$

$$\approx (N \ln N - N) - (N_{\uparrow} \ln N_{\uparrow} - N_{\uparrow}) - ((N - N_{\uparrow}) \ln (N - N_{\uparrow}) - (N - N_{\uparrow})) \quad (6)$$

$$= N \ln N - N_{\uparrow} \ln N_{\uparrow} - (N - N_{\uparrow}) \ln (N - N_{\uparrow}) \quad (7)$$

We can now get the temperature from

$$\frac{1}{T} = \frac{\partial S}{\partial U} \quad (8)$$

Using the chain rule, 2 and 1 we get

$$\frac{1}{T} = \frac{\partial S}{\partial N_{\uparrow}} \frac{\partial N_{\uparrow}}{\partial U} \quad (9)$$

$$\frac{\partial S}{\partial N_{\uparrow}} = k [-\ln N_{\uparrow} - 1 + \ln (N - N_{\uparrow}) + 1] \quad (10)$$

$$= k \ln \frac{N - N_{\uparrow}}{N_{\uparrow}} \quad (11)$$

$$= k \ln \frac{N + U/\mu B}{N - U/\mu B} \quad (12)$$

$$\frac{\partial N_{\uparrow}}{\partial U} = -\frac{1}{2\mu B} \quad (13)$$

$$\frac{1}{T} = -\frac{k}{2\mu B} \ln \frac{N + U/\mu B}{N - U/\mu B} \quad (14)$$

$$= \frac{k}{2\mu B} \ln \frac{N - U/\mu B}{N + U/\mu B} \quad (15)$$

Exponentiating both sides gives

$$e^{2\mu B/kT} = \frac{N - U/\mu B}{N + U/\mu B} \quad (16)$$

which can be solved for  $U$ :

$$\frac{U}{\mu B} [e^{2\mu B/kT} + 1] = N [1 - e^{2\mu B/kT}] \quad (17)$$

$$U = N\mu B \frac{1 - e^{2\mu B/kT}}{1 + e^{2\mu B/kT}} \quad (18)$$

$$= N\mu B \frac{e^{-\mu B/kT} - e^{\mu B/kT}}{e^{-\mu B/kT} + e^{\mu B/kT}} \quad (19)$$

$$= -N\mu B \tanh \frac{\mu B}{kT} \quad (20)$$

The magnetism is, from 3

$$M = N\mu \tanh \frac{\mu B}{kT} \quad (21)$$

Finally, the heat capacity at constant magnetic field is (using the derivative of the tanh function:  $d \tanh x / dx = \text{sech}^2 x$ ):

$$C_B = \frac{\partial U}{\partial T} \quad (22)$$

$$= -N\mu B \left[ \text{sech}^2 \frac{\mu B}{kT} \right] \left[ -\frac{\mu B}{kT^2} \right] \quad (23)$$

$$= \frac{Nk(\mu B/kT)^2}{\cosh^2(\mu B/kT)} \quad (24)$$

These analytic formulas give the curves we saw earlier using the numerical solution.

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