TWO-STATE PARAMAGNET: ANALYTIC SOLUTION

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 3.19.

We've looked at a two-state paramagnet and generated some curves involving entropy, energy and temperature using numerical methods. Here we'll derive the analytic solution.

This is a system of N magnetic dipoles which, when placed in a magnetic field B, align themselves so that their magnetic moment μ points either parallel or antiparallel to the field. The energy of a dipole that is aligned with the field is lower, and we'll call it $-\mu B$, so that the antiparallel dipole has energy $+\mu B$, and the total energy of the system is

$$U = \mu B\left(N_{\downarrow}-N_{\uparrow}\right)=\mu B\left(N-2N_{\uparrow}\right) \tag{1}$$

$$N_{\uparrow} = \frac{1}{2}\left(N-\frac{U}{\mu B}\right) \tag{2}$$

where an up arrow indicates parallel alignment and a down arrow antiparallel.

The net magnetization is then

$$M=\mu\left(N_{\uparrow}-N_{\downarrow}\right)=-\frac{U}{B} \tag{3}$$

The multiplicity of states is the same as a set of N coins with $N_{\uparrow}$ heads, so

$$\Omega=\binom{N}{N_{\uparrow}}=\frac{N!}{N_{\uparrow}!\left(N-N_{\uparrow}\right)!} \tag{4}$$

We begin the analytic solution by looking at the entropy directly, and using Stirling's approximation.

(5)
$$\frac{S}{k} = \ln\Omega$$
(6)
$$\approx (N\ln N - N) - \left(N_\uparrow \ln N_\uparrow - N_\uparrow\right) - \left(\left(N - N_\uparrow\right)\ln\left(N - N_\uparrow\right) - \left(N - N_\uparrow\right)\right)$$
(7)
$$= N\ln N - N_\uparrow \ln N_\uparrow - \left(N - N_\uparrow\right)\ln\left(N - N_\uparrow\right)$$

We can now get the temperature from

$$\frac{1}{T} = \frac{\partial S}{\partial U} \tag{8}$$

Using the chain rule, 2 and 1 we get

$$\begin{aligned}
\frac{1}{T} &= \frac{\partial S}{\partial N_\uparrow}\frac{\partial N_\uparrow}{\partial U} && (9)\\
\frac{\partial S}{\partial N_\uparrow} &= k\left[-\ln N_\uparrow - 1 + \ln\left(N - N_\uparrow\right) + 1\right] && (10)\\
&= k\ln\frac{N - N_\uparrow}{N_\uparrow} && (11)\\
&= k\ln\frac{N + U/\mu B}{N - U/\mu B} && (12)\\
\frac{\partial N_\uparrow}{\partial U} &= -\frac{1}{2\mu B} && (13)\\
\frac{1}{T} &= -\frac{k}{2\mu B}\ln\frac{N + U/\mu B}{N - U/\mu B} && (14)\\
&= \frac{k}{2\mu B}\ln\frac{N - U/\mu B}{N + U/\mu B} && (15)
\end{aligned}$$

Exponentiating both sides gives

$$e^{2\mu B/kT} = \frac{N - U/\mu B}{N + U/\mu B} \tag{16}$$

which can be solved for U:

$$\frac{U}{\mu B}\left[e^{2\mu B/kT}+1\right] = N\left[1-e^{2\mu B/kT}\right] \tag{17}$$

$$U = N\mu B\frac{1-e^{2\mu B/kT}}{1+e^{2\mu B/kT}} \tag{18}$$

$$= N\mu B\frac{e^{-\mu B/kT}-e^{\mu B/kT}}{e^{-\mu B/kT}+e^{\mu B/kT}} \tag{19}$$

$$= -N\mu B\tanh\frac{\mu B}{kT} \tag{20}$$

The magnetism is, from 3

$$M = N\mu\tanh\frac{\mu B}{kT} \tag{21}$$

Finally, the heat capacity at constant magnetic field is (using the derivative of the tanh function: $d\tanh x/dx = \mathrm{sech}^2 x$):

$$C_B = \frac{\partial U}{\partial T} \tag{22}$$

$$= -N\mu B\left[\mathrm{sech}^2\frac{\mu B}{kT}\right]\left[-\frac{\mu B}{kT^2}\right] \tag{23}$$

$$= \frac{Nk\left(\mu B/kT\right)^2}{\cosh^2\left(\mu B/kT\right)} \tag{24}$$

These analytic formulas give the curves we saw earlier using the numerical solution.

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