CARNOT CYCLE

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 4.5.

An ideal heat engine absorbs an amount Q_h from a hot reservoir at temperature T_h, converts an amount W of this heat to work and expels the remaining heat Q_c to a cold reservoir at tempearture T_c. From the first and second laws of thermodynamics, we can show that the maximum efficiency of such a heat engine is

$$e_{max} = 1 - \frac{T_c}{T_h} \tag{1}$$

In practice, real engines achieve considerably less than this efficiency, but is it possible to construct an engine that *does* achieve the maximum? It is, in principle, possible, and the path that such an engine must follow on a PV diagram is called the Carnot cycle.

The idea is to minimize the entropy generated at each stage. If the gas absorbs an amount Q_h from the hot reservoir, the entropy of the reservoir decreases by Q_h/T_h and the entropy of the gas increases by Q_h/T_{gas}. If $T_{gas} < T_h$, then the net entropy generated is greater than zero. To minimize this, we must therefore make T_{gas} as close to T_h as we can. We can't make them equal, since then no heat would flow from the reservoir to the gas, so we'll make T_{gas} infinitesimally less than T_h. Of course, this also means that it will take a very long time for the gas to absorb the heat, since such a small temperature difference means a very slow rate of heat flow.

Similarly, at the cold reservoir, to avoid extra entropy we make the temperature of the gas infinitesimally greater than T_c. In between the hot and cold reservoirs, we would like the heat flow to be zero, since this means no entropy flows at all in these stages.

Thus the two stages where the gas is in contact with either the hot or cold reservoir take place at a constant temperature (the temperature of the respective reservoir), so they are isothermal processes. The two stages of the cycle where the gas cools off and then warms up again take place with no heat transfer (so the temperature changes are achieved by changing the pressure and volume). Thus these two stages are adiabatic processes.

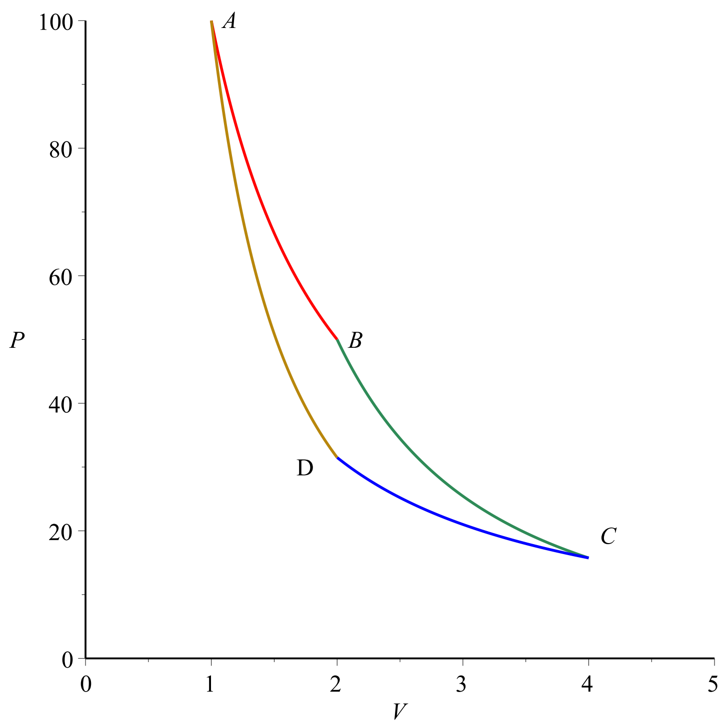
On a PV diagram, and isothermal curve obeys the relation

$$P=\frac{NkT}{V} \tag{2}$$

while an adiabatic curve obeys

$$P=\frac{K}{V^{\gamma}} \tag{3}$$

where K is a constant and $\gamma=(f+2)/f$ where f is the number of degrees of freedom of each molecule. For a monatomic ideal gas, $\gamma=\frac{5}{3}$. A Carnot cycle looks like this:



[The units are arbitrary. It's the shape of the curve that's important.] The gas traverses the cycle in a clockwise direction, with the red $A\rightarrow B$ and blue $C\rightarrow D$ legs being the isothermals, and the green $B\rightarrow C$ and yellow $D\rightarrow A$ legs the adiabats.

To prove that the Carnot cycle does in fact achieve the maximum efficiency, we can calculate the heats Q_h and Q_c. In the leg $A\rightarrow B$, the gas is in contact with the hot reservoir and undergoes an isothermal expansion. Since the temperature doesn't change, the energy of the gas remains constant, so the heat added goes entirely into the work of expansion of the gas. That is

$$Q_h = \int_{V_A}^{V_B} P\,dV \tag{4}$$

$$= NkT_h \int_{V_A}^{V_B} \frac{dV}{V} \tag{5}$$

$$= NkT_h \ln \frac{V_B}{V_A} \tag{6}$$

Similarly, the heat expelled to the cold reservoir along the leg $C \to D$ is

$$Q_c = NkT_c \ln \frac{V_C}{V_D} \tag{7}$$

[Note that Q_h and Q_c flow in opposite directions relative to the gas, although they both have the same sign.]

From 3 we have for the $B \to C$ leg

$$P_B V_B^\gamma = P_C V_C^\gamma \tag{8}$$

However, at point B, the adiabat joins the isothermal curve, so

$$P_B = \frac{NkT_B}{V_B} \tag{9}$$

and similarly for point C, so

$$NkT_B V_B^{\gamma-1} = NkT_C V_C^{\gamma-1} \tag{10}$$

$$\frac{T_B}{T_C} = \left(\frac{V_C}{V_B}\right)^{\gamma-1} \tag{11}$$

Applying the same logic to the $D \to A$ leg, and using $T_A = T_B$ and $T_C = T_D$ we get

$$\frac{T_A}{T_D} = \frac{T_B}{T_C} = \left(\frac{V_D}{V_A}\right)^{\gamma-1} \tag{12}$$

Comparing the last two equations we find that

$$\frac{V_D}{V_A} = \frac{V_C}{V_B} \tag{13}$$

or, rearranging:

$$\frac{V_B}{V_A} = \frac{V_C}{V_D} \tag{14}$$

Inserting this into 6 and 7 and taking the ratio we get

$$\frac{Q_c}{Q_h} = \frac{NkT_c \ln \frac{V_C}{V_D}}{NkT_h \ln \frac{V_B}{V_A}} = \frac{T_c}{T_h} \tag{15}$$

Thus the efficiency of a Carnot cycle is

$$e = 1 - \frac{Q_c}{Q_h} = 1 - \frac{T_c}{T_h} \tag{16}$$

QED.

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