CARNOT ENGINE - A REALISTIC VERSION

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 4.6.

The main problem with an engine that follows a Carnot cycle is that the two isothermal stages in the cycle proceed very slowly, since we are attempting to transfer heat between two systems that are almost at the same temperature. One way of making the cycle go a bit faster is to make the temperature of the working substance significantly different from that of the reservoir where it absorbs, and later expels, heat. That is, if the system absorbs heat Q_h from a hot reservoir at temperature T_h, then the temperature of the working substance (typically a gas) when it absorbs heat is $T_{hw} < T_h$. Similarly, at the other isothermal stage where heat Q_c is expelled to the cold reservoir at temperature T_c, the temperature of the gas is $T_{cw} > T_c$.

To make things simple, we'll assume that the rate of heat transfer is the same at both the hot and cold reservoirs, and is proportional to the temperature difference between the gas and the reservoir. That is

$$\frac{Q_h}{\Delta t} = K\left(T_h - T_{hw}\right) \tag{1}$$

$$\frac{Q_c}{\Delta t} = K\left(T_{cw} - T_c\right) \tag{2}$$

where K is a constant and Δt is taken to be the same for both cases (that is, the durations of both isothermal stages in the cycle are the same). From this, we get the relation

$$\frac{Q_h}{T_h - T_{hw}} = \frac{Q_c}{T_{cw} - T_c} \tag{3}$$

If the only entropy that is created in the cycle is along the two isothermal stages (no entropy is generated along the adiabatic stages) then, since the state of the engine is the same at the end of the cycle as it was at the start, the gas must have expelled exactly the same amount of entropy when expelling heat to the cold reservoir as it absorbed when absorbing heat from the hot reservoir. That is

$$\frac{Q_h}{T_{hw}} = \frac{Q_c}{T_{cw}} \tag{4}$$

so

$$Q_c = Q_h \frac{T_{cw}}{T_{hw}} \tag{5}$$

Combining this with 3 gives

$$\frac{1}{T_h - T_{hw}} = \frac{T_{cw}}{T_{hw}(T_{cw} - T_c)} \tag{6}$$

$$T_{cw} = \frac{T_c T_{hw}}{2T_{hw} - T_h} \tag{7}$$

If the time required for the two adiabatic steps is much less than that for the two isothermal steps, we can work out the power output of the engine. The work is produced over a time interval of $2\Delta t$ and is

$$\mathcal{P} = \frac{W}{2\Delta t} = \frac{1}{2\Delta t}(Q_h - Q_c) \tag{8}$$

$$= \frac{K}{2}(T_h + T_c - T_{hw} - T_{cw}) \tag{9}$$

$$= \frac{K}{2}\left(T_h + T_c - T_{hw} - \frac{T_c T_{hw}}{2T_{hw} - T_h}\right) \tag{10}$$

We can maximize the power output for given values of T_h and T_c by varying T_{hw}. Taking the derivative we get

$$\frac{d\mathcal{P}}{dT_{hw}} = \frac{K}{2}\left[-1 - \frac{T_c}{2T_{hw} - T_h} + \frac{2T_c T_{hw}}{(2T_{hw} - T_h)^2}\right] = 0 \tag{11}$$

This can be solved for T_{hw} by multiplying through by $(2T_{hw} - T_h)^2$ and expanding the terms in the numerator. This results in

$$\frac{K\left(-4T_{hw}^2 + 4T_h T_{hw} + T_c T_h - T_h^2\right)}{2(2T_{hw} - T_h)^2} = 0 \tag{12}$$

Solving the quadratic equation and taking the positive root gives

$$T_{hw} = \frac{1}{2}\left(T_h + \sqrt{T_h T_c}\right) \tag{13}$$

Substituting this into 7 gives

$$T_{cw} = \frac{1}{2}\left(T_c + \sqrt{T_h T_c}\right) \tag{14}$$

To find the efficiency we have, using 4

$$\begin{aligned}
e &= 1-\frac{Q_c}{Q_h} && (15)\\
&= 1-\frac{T_{cw}}{T_{hw}} && (16)\\
&= 1-\frac{T_c+\sqrt{T_hT_c}}{T_h+\sqrt{T_hT_c}} && (17)\\
&= 1-\frac{\left(T_c+\sqrt{T_hT_c}\right)\left(T_h-\sqrt{T_hT_c}\right)}{T_h^2-T_hT_c} && (18)\\
&= 1-\frac{T_hT_c+(T_h-T_c)\sqrt{T_hT_c}-T_hT_c}{T_h(T_h-T_c)} && (19)\\
&= 1-\sqrt{\frac{T_c}{T_h}} && (20)
\end{aligned}$$

For a coal-fired steam turbine with $T_h = 600^\circ$ C $= 873$ K and $T_c = 25^\circ$ C $=$ 298 K, this gives an efficiency of

$$e = 0.416 \tag{21}$$

This is very close to the actual efficiency of about 40% for a real coal-burning power plant. The 'ideal' Carnot efficiency for these temperatures is

$$e = 1-\frac{T_c}{T_h} = 0.659 \tag{22}$$