STIRLING ENGINE

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 4.21.

A Stirling engine is a heat engine that works by absorbing heat from a hot reservoir, doing work, and expelling waste heat to a cold reservoir. In theory, it can approach the optimal efficiency of a Carnot engine. The engine follows four steps, as shown in the PV diagram:
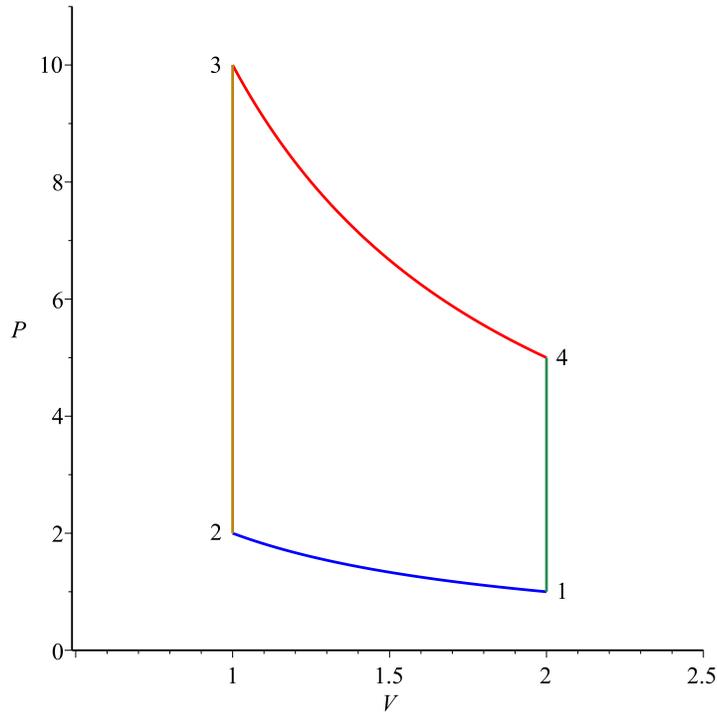
The engine contains two cylinders with pistons, with one cylinder connected to the hot reservoir and the other to the cold reservoir. The two cylinders are connected internally by means of a regenerator, which serves as a heat sink. Starting a point 1, the cold piston moves in, compressing the gas isothermally to point 2. At point 2, the gas is passed through the regenerator where it absorbs heat at constant volume. From point 3, the hot piston moves out doing work while the gas expands isothermally to point 4. At point 4, the gas is passed back through the regenerator where it expels an

amount of heat at constant volume so that its pressure returns to its starting value.

In effect, the regenerator acts as an internal reservoir that is capable of both absorbing and emitting heat. The precise mechanism by which it does this won't divert us here, but those interested can look at the Wikipedia article which describes it in some depth.

To analyze the engine, we'll suppose first that the regenerator isn't present, which means that all the absorbed heat must come from the hot reservoir and all the expelled heat goes to the cold reservoir. Heat is absorbed along the yellow and red curves. For the yellow path (constant volume), we have, from the equipartition theorem

$$Q_{23} = \frac{f}{2} Nk (T_3 - T_2) \tag{1}$$

where f is the number of degrees of freedom per molecule.

For the red path, since it's an isothermal, we have, using the ideal gas law $PV = NkT$:

$$Q_{34} = \int_{V_2}^{V_1} P \, dV = NkT_3 \ln \frac{V_1}{V_2} \tag{2}$$

Thus the total absorbed heat is

$$Q_h = \frac{f}{2} Nk (T_3 - T_2) + NkT_3 \ln \frac{V_1}{V_2} \tag{3}$$

A similar calculation along the green and blue paths gives us the total heat expelled:

$$Q_c = \frac{f}{2} Nk (T_3 - T_2) + NkT_2 \ln \frac{V_1}{V_2} \tag{4}$$

Note that the only difference between Q_h and Q_c is that the first contains T_3 in the last term, while the second contains T_2. The efficiency is therefore

$$e = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{\frac{f}{2}(T_3 - T_2) + T_2 \ln \frac{V_1}{V_2}}{\frac{f}{2}(T_3 - T_2) + T_3 \ln \frac{V_1}{V_2}} \tag{5}$$

Now suppose the regenerator is present and works perfectly, so that all the heat absorbed along the yellow edge is obtained from it, and all heat expelled along the green edge is absorbed by it. This is theoretically possible, since the heat absorbed along the yellow edge is equal to the heat expelled along the green edge, so energy within the regenerator is conserved. In that case

$$Q_h = NkT_3 \ln \frac{V_1}{V_2} \tag{6}$$

$$Q_c = NkT_2 \ln \frac{V_1}{V_2} \tag{7}$$

$$e = 1 - \frac{NkT_2 \ln \frac{V_1}{V_2}}{NkT_3 \ln \frac{V_1}{V_2}} \tag{8}$$

$$= 1 - \frac{T_2}{T_3} \tag{9}$$

which is the maximum possible efficiency (that of a Carnot engine).

The presence of the extra term $\frac{f}{2}(T_3 - T_2)$ in the numerator and denominator in 5 reduces the efficiency below this ideal value. I would imagine that the limiting factor in a Stirling engine would be how quickly the regenerator can absorb and expel heat. In a Carnot engine, the temperatures of the working substance are only infinitesimally different from the temperatures of the hot and cold reservoirs along the isothermal paths where heat is exchanged, which results in the engine being very slow. The gas in a Stirling engine exchanges heat with the regenerator as it passes through it (the regenerator is typically composed of a system with a large surface area, such as a wire mesh) so if that is slow, then the engine is less practical.