MAGNETIC SYSTEMS IN THERMODYNAMICS

We can derive thermodynamic identities for a magnetic system in which pressure and volume are constant. In such a system, work is done by an external electrical power source such as a battery, with the work resulting in changes to the magnetization and magnetic field present in the system, rather than changes in pressure or volume.

As an example, consider a solenoid with a (very long, so we can approximate the field produced as equivalent to that produced by an infinite solenoid) length $L$ and $n$ turns per unit length. Suppose that the interior of the solenoid is filled with a cylinder of magnetic material with a magnetization (magnetic dipole moment per unit volume) of $M$. By applying Ampère’s law to a loop with one edge inside and the opposite edge outside the solenoid, and using the fact that the magnetic field outside a solenoid is zero (see Griffiths referenced above, example 5.9), we have

$$\int \left( \frac{1}{\mu_0} B - M \right) \cdot d\ell = I_f$$

(1)

where $I_f$ is the free current enclosed by the loop. By choosing the loop edge to have a unit length we have $I_f = nI$, where $I$ is the current in the wire. The integrand is defined to be the auxiliary field $H$:

$$H \equiv \frac{1}{\mu_0} B - M$$

(2)

Since both $B$ and $M$ are parallel to the axis of the solenoid, the integrand is non-zero only on the edge of the loop inside the solenoid and we have

$$H = nI$$

(3)

I’m using $H$ to represent magnetic field to distinguish it from $H$, which represents enthalpy.
This is true for a steady current, but now suppose we vary the current to change the magnetic field strength $B$. According to Faraday’s law, a change in the magnetic flux $\Phi$ through a loop causes a back-emf $\mathcal{E}$ to be produced which opposes the change in flux, according to

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

(4)

The work done by the battery is therefore the work required to overcome this back-emf and thus keep the current flowing at its new value. The total magnetic flux through the solenoid is the flux through a single turn (which is $BA$, where $A$ is the cross-sectional area of the solenoid) multiplied by the total number of turns, which is $nL$. Therefore

$$\frac{d\Phi}{dt} = nAL\frac{dB}{dt}$$

(5)

The power (work per unit time) generated by the battery is the voltage times the current, which is $-\mathcal{E}I$, (negative because the battery’s voltage is opposite to the back-emf) so the work done in time $dt$ is

$$dW = -\mathcal{E}I \, dt$$

(6)

$$= \left( nAL\frac{dB}{dt} \right) \left( \frac{\mathcal{H}}{n} \right) \, dt$$

(7)

$$= AL\mathcal{H} \, dB$$

(8)

$$= V\mathcal{H} \, dB$$

(9)

where $V = AL$ is the volume of the solenoid.

From (2), we have

$$dB = \mu_0 \left( d\mathcal{H} + dM \right)$$

(10)

$$dW_{tot} = \mu_0 V \left( \mathcal{H} \, d\mathcal{H} + \mathcal{H} \, dM \right)$$

(11)

The first term can be written as

$$\mu_0 V \mathcal{H} \, d\mathcal{H} = \frac{\mu_0 V}{2} d\left( \mathcal{H}^2 \right)$$

(12)

In a vacuum, the energy density in a magnetic field is

$$E_m = \frac{1}{2\mu_0} B^2$$

(13)

and $B = \mu_0 \mathcal{H}$ so

$$E_m = \frac{\mu_0}{2} \mathcal{H}^2$$

(14)
Thus, (12) is the change in the magnetic field energy in the entire solenoid, assuming there is a vacuum inside the solenoid. [Actually, in a linear magnetic material \( B = \mu H \), where \( \mu \) is the permeability of the material, and it is only in a vacuum that \( \mu = \mu_0 \). Thus, (14) is the magnetic energy density in a vacuum.]

The second term in (11) is the work required to change the magnetization of the sample inside the solenoid:

\[
\frac{dW}{dM} = \mu_0 H V dM = \mu_0 H dM \tag{15}
\]

where \( M \equiv V M \) is the total magnetization of the sample.

Assuming that only the energy and magnetization change, we can work out a thermodynamic identity for magnetic matter. By analogy with the earlier derivation, we can imagine a system in which the entropy changes by either the energy \( U \) changing or the magnetization \( M \) changing. The entropy change due to an increase in energy at temperature \( T \) is

\[
dS_U = \frac{dU}{T} \tag{17}
\]

How does the entropy change when the magnetization \( M \) increases? A larger magnetization usually means that the dipoles in the sample are more ordered, so we’d expect an increase in \( M \) to cause a decrease in \( S \). Thus we’d expect

\[
\left( \frac{\partial S}{\partial M} \right)_U < 0 \tag{18}
\]

From (16) we’d like \( \left( \frac{\partial S}{\partial M} \right)_U \) to be something times \( H \). To find the 'something', we can look at the units. \( S \) has units of \( \text{J} \text{K}^{-1} \). From (2) \( M \) has units of \( \text{a(volume)} \times B/\mu_0 \) which works out to \( \text{m}^3 \text{(Tesla)} \text{(J m}^{-1}\text{Amp}^{-2})^{-1} = \text{m}^3 \text{(J Amp}^{-1}\text{m}^{-2}) \text{(J}^{-1}\text{m Amp}^2) = \text{m}^2\text{Amp} \). The derivative \( \left( \frac{\partial S}{\partial M} \right)_U \) therefore has units of \( \text{J m}^{-2}\text{Amp}^{-1}\text{K}^{-1} \). Multiplying this by \( T/\mu_0 \) gives a quantity with units of \( \text{Amp m}^{-1} \) which are the units of \( H \), so we propose

\[
\frac{T}{\mu_0} \left( \frac{\partial S}{\partial M} \right)_U = -H \tag{19}
\]

The total entropy change is therefore

\[
dS = dS_U + dS_M \tag{20}
\]

\[
= \frac{dU}{T} - \frac{\mu_0 H}{T} dM \tag{21}
\]
This gives a thermodynamic identity of

$$dU = T \, dS + \mu_0 H \, dM$$

(22)

As we might expect, the last term is just the work done on the system as given by [16].

By analogy with the definition of enthalpy for a pressure-volume system, the enthalpy here is

$$H = U - \mu_0 H M$$

(23)

This is the energy required to create a magnetic system from scratch, with an internal energy $U$ and magnetization $M$ in a field $H$. Presumably the second term is negative, since aligning the dipoles with the external field reduces the energy of the system.

The corresponding thermodynamic identity is

$$dH = dU - \mu_0 H dM - \mu_0 M dH$$

(24)

$$= T \, dS - \mu_0 M dH$$

(25)

The Helmholtz free energy is still defined as

$$F = U - TS$$

(26)

since there is no reference to pressure or volume. The thermodynamic potential is

$$dF = dU - T \, dS - S \, dT$$

(27)

$$= -S \, dT + \mu_0 H \, dM$$

(28)

The Gibbs free energy is thus defined as

$$G = U - TS - \mu_0 H M$$

(29)

$$= F - TS$$

(30)

Its thermodynamic identity is

$$dG = dF - \mu_0 H \, dM - \mu_0 M \, dH$$

(31)

$$= -S \, dT - \mu_0 M \, dH$$

(32)