

WET ADIABATIC LAPSE RATE

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Reference: Daniel V. Schroeder, *An Introduction to Thermal Physics*, (Addison-Wesley, 2000) - Problem 5.45.

Earlier, we worked out the dry adiabatic lapse rate, which is the temperature gradient in dry air at which convection begins, that is, it is the point at which air near the surface achieves a density low enough that it starts to rise adiabatically. However, in the real atmosphere there will always be some water vapour, and as the air rises and cools, some of this water vapour will condense out, forming clouds. When water vapour condenses to liquid, it releases an amount of heat given by the latent heat of vapourization L , causing a decrease in the rate at which cooling occurs with height. We can get an estimate of this revised rate of cooling, called the *wet adiabatic lapse rate* (sometimes the *moist adiabatic lapse rate*) by combining some of our earlier results.

First, we can use the first law of thermodynamics (conservation of energy) to include the effect of condensation. Suppose we have a mass of air containing n moles of air molecules and n_w moles of water vapour (we'll assume that $n_w \ll n$ in what follows). If an amount dn_w of water vapour condenses, it releases an amount of heat $L dn_w$. As the air rises, it also expands by a volume dV , doing work $P dV$, so the net energy change of the air mass is

$$(1) \quad dU = -P dV - L dn_w$$

The first negative sign is because the gas does work (and hence loses energy) as it expands, while the second negative sign is because a decrease in n_w (that is, $dn_w < 0$) releases heat, thus increasing dU .

We want to convert this to a formula for the change in temperature dT in terms of dP and dn_w . From the equipartition theorem, the energy of the air mass is

$$(2) \quad U = \frac{f}{2} nRT$$

where f is the number of degrees of freedom of an air molecule, which is usually taken to be $f = 5$. We therefore have

$$(3) \quad dU = \frac{5}{2}nR dT$$

We now need to convert the $P dV$ term. My first thought was that, since the gas is supposed to be expanding adiabatically, we can use the relation

$$(4) \quad PV^\gamma = K$$

where $\gamma = (f + 2)/f = 7/5$ and K is a constant. Taking differentials gives

$$(5) \quad dV = -\frac{V}{\gamma P}dP = -\frac{5V}{7P}dP$$

and plugging all this back into 1 gives

$$(6) \quad \frac{5}{2}nR dT = \frac{5V}{7}dP - L dn_w$$

$$(7) \quad dT = \frac{2V}{7nR}dP - \frac{2}{5} \frac{L}{nR}dn_w$$

Applying the ideal gas law $PV = nRT$ to the first term, we get

$$(8) \quad dT = \frac{2T}{7P}dP - \frac{2}{5} \frac{L}{nR}dn_w$$

This does not agree with Schroeder's result, which has $\frac{2}{7}$ instead of $\frac{2}{5}$ in the last term.

We can get Schroeder's answer if instead of using the adiabatic relation 4, we use the ideal gas law and take differentials:

$$(9) \quad P dV + V dP = nR dT$$

$$(10) \quad P dV = nR dT - V dP$$

We now get

$$(11) \quad \frac{5}{2}nR dT = -nR dT + V dP - L dn_w$$

$$(12) \quad \frac{7}{2}nR dT = V dP - L dn_w$$

$$(13) \quad = \frac{nRT}{P}dP - L dn_w$$

$$(14) \quad dT = \frac{2T}{7P}dP - \frac{2}{7} \frac{L}{nR}dn_w$$

I'm not entirely sure why the adiabatic relation gives the wrong answer, although it could have something to do with the fact that, strictly speaking, the process isn't adiabatic, since the condensing vapour injects some heat into the air mass.

In any case, we can proceed with the derivation of the wet adiabatic lapse rate. The next step is to assume that the air remains saturated (100% humidity) at all stages of the expansion. In this case, the ratio n_w/n satisfies

$$(15) \quad \frac{n_w}{n} = \frac{P_v(T)}{P}$$

where P_v is the vapour pressure at which condensation occurs at temperature T . Thus, $n_w = n_w(P, T)$ is a function of total air pressure P and temperature T at a given height z . We can now take the total derivative to get

$$(16) \quad \frac{dn_w}{dz} = \frac{\partial n_w}{\partial P} \frac{dP}{dz} + \frac{\partial n_w}{\partial T} \frac{dT}{dz}$$

The partial derivatives are

$$(17) \quad \frac{\partial n_w}{\partial P} = -\frac{nP_v}{P^2}$$

$$(18) \quad \frac{\partial n_w}{\partial T} = \frac{n}{P} \frac{dP_v}{dT}$$

So we get

$$(19) \quad \frac{dn_w}{dz} = -\frac{nP_v}{P^2} \frac{dP}{dz} + \frac{n}{P} \frac{dP_v}{dT} \frac{dT}{dz}$$

From the Clausius-Clapeyron equation, we have

$$(20) \quad \frac{dP_v}{dT} = \frac{L}{T\Delta V}$$

where L is the latent heat of vapourization per mole, and ΔV is the difference in volume between the gas and liquid phases. Since the volume of the liquid is negligible relative to the volume of the gas, we can approximate ΔV using the ideal gas law:

$$(21) \quad \Delta V = \frac{n_w RT}{P_v} = \frac{RT}{P_v}$$

[Note that $n_w = 1$ here since L in 20 is the latent heat *per mole*.] This gives

$$(22) \quad \frac{dP_v}{dT} = \frac{LP_v}{RT^2}$$

Therefore, 19 becomes

$$(23) \quad \frac{dn_w}{dz} = -\frac{nP_v}{P^2} \frac{dP}{dz} + \frac{nLP_v}{PRT^2} \frac{dT}{dz}$$

Returning to 14, we can divide both sides by dz and substitute in 23 to get

$$(24) \quad \frac{dT}{dz} = \frac{2T}{7P} \frac{dP}{dz} - \frac{2L}{7nR} \left[-\frac{nP_v}{P^2} \frac{dP}{dz} + \frac{nLP_v}{PRT^2} \frac{dT}{dz} \right]$$

$$(25) \left[1 + \frac{2P_v}{7P} \left(\frac{L}{RT} \right)^2 \right] \frac{dT}{dz} = \frac{2}{7} \left[\frac{T}{P} + \frac{LP_v}{RP^2} \right] \frac{dP}{dz}$$

We can now use the barometric equation on the RHS:

$$(26) \quad \frac{dP}{dz} = -\frac{mg}{kT} P$$

$$(27) \quad = -\frac{Mg}{RT} P$$

where m is the mass of one air molecule and M is the mass of a mole of air molecules. This gives

$$(28) \quad \left[1 + \frac{2P_v}{7P} \left(\frac{L}{RT} \right)^2 \right] \frac{dT}{dz} = -\frac{2Mg}{7R} \left[1 + \frac{P_v L}{P RT} \right]$$

$$(29) \quad \frac{dT}{dz} = -\frac{2Mg}{7R} \left[1 + \frac{P_v L}{P RT} \right] \left[1 + \frac{2P_v}{7P} \left(\frac{L}{RT} \right)^2 \right]^{-1}$$

If we plug in some numbers we can see the effect of condensation on the lapse rate. We'll use the earlier expression for P_v and other values from Schroeder's book:

$$\begin{aligned}
 (30) \quad P_v &= Ke^{-L/RT} \\
 (31) \quad K &= 1.63 \times 10^{11} \text{ Pa} \\
 (32) \quad m &= 4.81 \times 10^{-26} \text{ kg} \\
 (33) \quad M &= N_A m = 0.029 \text{ kg} \\
 (34) \quad L &= 4.399 \times 10^4 \\
 (35) \quad R &= 8.314 \\
 (36) \quad g &= 9.8
 \end{aligned}$$

[The last 3 quantities are in SI units.]

At a temperature of $T = 25^\circ \text{C} = 298 \text{ K}$ and pressure of 1 bar (10^5 Pa), we have

$$(37) \quad \frac{dT}{dz} = -0.00395 \text{ K m}^{-1}$$

or about 4 K per kilometre. The dry lapse rate is around 9.75 K per kilometre, so condensation has a considerable effect.

At the lower temperature of $0^\circ \text{C} = 273 \text{ K}$ and pressure of 1 bar we have

$$(38) \quad \frac{dT}{dz} = 6.5 \text{ K km}^{-1}$$

Presumably this is because at lower temperatures the air can hold less water vapour, so the air is dryer, which means less vapour is available for condensation.