SUBSPACES AND DIRECT SUMS

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References: edX online course MIT 8.05.1x Week 3.
Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 1.

Subspaces. Having defined a general vector space, we can now consider subspaces of a vector space. Put simply, a subspace is a subset of a vector space that is itself a vector space. That is, the elements of the subspace must satisfy all the conditions of a vector space, which are

(1) A vector space is a set V with two operations, addition and scalar multiplication, defined on the set.

(2) The addition property is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$. Note that this definition implies completeness, in the sense that every sum of two vectors in V must also be in V. This definition includes the traditional notion of vector addition in 2-d or 3-d space (that is, where a vector is represented by an arrow, and vector addition is performed by putting the tail of the second vector onto the head of the first and drawing the resulting vector as the sum), but vector addition is much more general than that.

(3) Scalar multiplication means that we can take an ordinary number λ from some field $\mathbb{F}$ (in quantum theory, $\mathbb{F}$ will always be either the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$) and define a function in which the vector obtained by multiplying an existing vector v by λ gives another vector $\lambda v \in V$. Note that again, completeness is implied by this definition: every vector λv obtained through scalar multiplication must also be in the space V.

(4) Addition is commutative, so that $u + v = v + u$.

(5) Addition and scalar multiplication are associative, so that $(u + v) + w = u + (v + w)$ and $(ab) v = a (bv)$, where $u, v, w \in V$ and $a, b \in \mathbb{F}$.

(6) There is an additive identity element $0 \in V$ such that $v + 0 = v$ for all $v \in V$. Note that here 0 is a vector, not a scalar. In practice, there is also a zero scalar number which is also denoted by 0, so we need to rely on the context to tell whether 0 refers to a vector or a number. Usually this isn't too hard.

(7) Every vector $v \in V$ has an additive inverse $w \in V$ with the property that $u + w = 0$. The additive inverse of v is written as $-v$ and $w - v$ is defined to be $w + (-v)$.

(8) There is a (scalar) multiplicative identity number 1 with the property that $1v = v$ for all $v \in V$.

(9) Scalar multiplication is distributive, in the sense that $a(u+v) = au + av$ and $(a+b)v = av + bv$ for all $a, b \in \mathbb{F}$ and all $u, v \in V$.

Since a subset will automatically satisfy all the conditions except possibly for numbers 2, 3 and 6, we need test only these three conditions to verify that a subset is a subspace. In particular, we need to check that

- The subset is closed under addition (satisfying property 2 above).
- The subset is closed under scalar multiplication (satisfying property 3).
- The subset contains the additive identity element 0 (satisfying property 6).

As an example, suppose we start with a vector space consisting of all the vectors pointing to locations in 3-d space (here, a 'vector' is the traditional line with an arrow on the end). Any subspace of this vector space must contain the origin (the additive identity). One possible subspace is just the origin on its own, since it satisfies all the above properties.

Any other subspace must be infinite in extent, since multiplication by a scalar can increase any non-zero to an arbitrarily large value. Thus any finite 3-d volume cannot be a subspace, even if it includes the origin. The possible subspaces are all planes that contain the origin (giving 2-d subspaces), and all lines that contain the origin (1-d subspaces).

Given any two subspaces U_1 and U_2, their intersection $U_1 \cap U_2$ is also a subspace. Being a physicist, I'm not going to give a rigorous proof of this, but the argument would go something like this. Any subspace is closed under both addition and scalar multiplication, so if both U_1 and U_2 contain the vectors u and v, they must both also contain all vectors of the form $au + bv$, where $a, b \in \mathbb{F}$. Thus their intersection will also contain all these vectors, making $U_1 \cap U_2$ a subspace.

Direct sums. A vector space V can be written as a direct sum of subspaces $U_1, \ldots, U_m$, written as

$$V = U_1 \oplus \ldots \oplus U_m \tag{1}$$

provided that any vector $v \in V$ can be written uniquely as

$$v = u_1 + \ldots + u_m \tag{2}$$

where $u_i \in U_i$. That is, any vector can be written as a sum consisting of one vector from each subspace. Note that since all subspaces contain the zero vector 0, one or more of the u_i could be 0. [Axler uses an ordinary plus sign + to denote a direct sum.]

The subspaces U_i in a direct sum cannot overlap (apart from containing 0). That is $U_i \cap U_j = \{0\}$ if $i \neq j$. We can see this since if we consider some vector $w \in U_i \cap U_j$, then $-w \in U_i \cap U_j$ (since $-w$ is the additive inverse of w; see condition 7 above). Thus the following two decompositions of a vector v are both valid:

$$v = u_1 + \ldots + u_i + \ldots + u_j + u_m \tag{3}$$
$$v = u_1 + \ldots + (u_i + w) + \ldots + (u_j - w) + u_m \tag{4}$$

Thus the decomposition isn't unique unless $U_i \cap U_j = \{0\}$ if $i \neq j$.

Another way of saying this is that the only way of writing 0 as a decomposition is if $u_i = 0$ for all i. (Since that is a valid decomposition of 0, and it must be unique.)

As an example, the space of 3-d points considered above can be decomposed into one plane and one line not in that plane (for example, the xy plane and the z axis), or into 3 non-coplanar lines (for example, the x, y and z axes).

As another example, consider the space of polynomials $p(z)$ of degree N. We can decompose this into a subspace of polynomials containing only even powers of z and another subspace containing only odd powers of z. Both subspaces contain 0 (obtained by setting all the coefficients to zero), and any general polynomial of degree N can be formed from the sum of one polynomial containing even powers and another containing odd powers.

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