VECTOR SPACES: SPAN, LINEAR INDEPENDENCE AND BASIS

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References: edX online course MIT 8.05.1x Week 3.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 2.

Here, we investigate the ideas of the span of a vector space and see how this leads to the idea of linear independence of a set of vectors. I'll summarize the main definitions and results here for future use; a more complete explanation together with some examples is given in Axler's book, Chapter 2.

Span of a list of vectors. A *list* of vectors is just a subset of the vectors in a vector space, with the condition that the number of vectors in the subset is finite. The set of all linear combinations of the vectors $(v_1,\ldots,v_m)$ in a list is called the *span* of that list. Since a general linear combination has the form

$$v=\sum_{i=1}^{m}a_iv_i \tag{1}$$

where $a_i\in\mathbb{F}$ (recall that the field $\mathbb{F}$ is always taken to be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$), the span of a list itself forms a vector space which is a subspace of the original vector space. One result we can show is

Theorem 1. *The span of a list of vectors in a vector space V is the smallest subspace of V containing all the vectors in the list.*

Proof. Let the list be $L\equiv(v_1,\ldots,v_m)$. Then $S\equiv\operatorname{span}(v_1,\ldots,v_m)$ is a subspace since it contains the zero vector if all a_is are zero in 1, and since it contains all linear combinations of the list, it is closed under addition and scalar multiplication.

The span S contains all $v_j\in L$ (just set $a_j=\delta_{ij}$ in 1). Now if we look at a subspace of V that contains all the v_is, it must also contain every vector in the span S, since a subspace must be closed under addition and scalar multiplication. Thus S is the smallest subspace of V that contains all the vectors in L. $\square$

If $S \equiv \text{span}(v_1, \ldots, v_m) = V$, that is, the span of a list is the same as the original vector space, then we say that $(v_1, \ldots, v_m)$ spans V. This leads to the definition that a vector space is called *finite-dimensional* if it is spanned by some list of vectors. (Remember that all lists are finite in length!) A vector space that is not finite-dimensional is called (not surprisingly) *infinite-dimensional*.

Linear independence. Suppose a list $(v_1, \ldots, v_m) \in V$ and v is a vector such that $v \in \text{span}(v_1, \ldots, v_m)$. This means that v is a linear combination of $(v_1, \ldots, v_m)$, so that 1 is true. However, using only the definitions above, there is no guarantee that there is only one choice for the scalars a_i that satisfies 1. We might also have, for example

$$v = \sum_{i=1}^{m} c_i v_i \tag{2}$$

where $c_i \neq a_i$. This means that we can write the zero vector as

$$0 = \sum_{i=1}^{m} (a_i - c_i) v_i \tag{3}$$

Now, if the *only* way we can satsify this equation is to require that $a_i = c_i$ for all i, then we say that the list $(v_1, \ldots, v_m)$ is *linearly independent*. (For completeness, the empty list (containing no vectors) is also declared to be linearly independent.) By reversing the above argument, we see that if the list $(v_1, \ldots, v_m)$ is linearly independent, then there is only one set of scalars a_i such that 1 is satisfied. In other words, any vector $v \in \text{span}(v_1, \ldots, v_m)$ has only one representation as a linear combination of the vectors in the list.

A list that is not linearly independent is, again not surprisingly, defined to be *linearly dependent*. This leads to the linear dependence lemma:

Lemma 2. *Suppose $(v_1, \ldots, v_m)$ is a linearly dependent list in V. Then there exists some $j \in \{1, 2, \ldots, m\}$ such that*

(a) $v_j \in span\left(v_1, \ldots, v_{j-1}\right)$;

(b) if v_j is removed from the list $(v_1, \ldots, v_m)$, the span of the remaining list, containing $m-1$ vectors, equals the span of the original list.

Proof. Because $(v_1, \ldots, v_m)$ is linearly dependent, we can write

$$\sum_{i=1}^{m} a_i v_i = 0 \tag{4}$$

where not all of the a_is are zero. Suppose j is the largest index where $a_j \neq 0$. Then we can divide through by a_j to get

$$v_j = -\frac{1}{a_j}\sum_{i=1}^{j-1} a_i v_i \tag{5}$$

Thus v_j is a linear combination of other vectors in the list, which proves part (a). Part (b) follows from the fact that we can represent any vector $u \in \operatorname{span}(v_1, \dots, v_m)$ as

$$u = \sum_{i=1}^{m} a_i v_i \tag{6}$$

We can replace v_j in this sum by 5, so u can be written as a linear combination of all the vectors in the list $(v_1, \dots, v_m)$ except for v_j. Thus (b) is true. □

We can use this lemma to prove the main result about linearly independent lists:

Theorem 3. *In a finite-dimensional vector space V, the length of every linearly independent list is less than or equal to the length of every list that spans V.*

Proof. Suppose the list $A \equiv (u_1, \dots, u_m)$ is linearly independent in V, and suppose another list $B \equiv (w_1, \dots, w_n)$ spans V. We want to prove that $m \leq n$.

Since B already spans V, if we add any other vector from V to the list B, we will get a linearly dependent list, since this newly added vector can, by the definition of a span, be expressed a linear combination of the vectors in B. In particular, if we add u_1 from the list A to B, then the list $(u_1, w_1, \dots, w_n)$ is linearly dependent. By the linear independence lemma above, we can therefore remove one of the w_is from B so that the remaining list still spans V, and contains n vectors. For the sake of argument, let's say we remove w_n (we can always order the w_is in the list so that the element we remove is at the end). Then we're left with the revised list $B_1 = (u_1, w_1, \dots, w_{n-1})$.

We can repeat this process m times, each time adding the next element u_i from list A and removing the last w_i. Because of the linear dependence lemma, we know that there must always be a w_i that can be removed each time we add a u_i, so there must be at least as many w_is as u_is. In other words, $m \leq n$ which is what we wanted to prove. □

This theorem can be used to show easily that any list of more than n vectors in n-dimensional space cannot be linearly independent, since we know that we can span n-dimensional space with n vectors (for example, the 3 coordinate axes in 3-d space). Conversely, since we can find a list of n vectors in n-dimensional space that is linearly independent, any list of fewer than n vectors cannot span n-dimensional space.

Basis of a finite-dimensional vector space. A basis of a finite-dimensional vector space is defined to be a list that is both linearly independent and spans the space. The *dimension* of the vector space is defined to be the length of a basis list. For example, in 3-d space, the list $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis, and since the length is 3, the dimension of the vector space is also 3. Any proper subset (that is, a subset with fewer than 3 members) of this basis is also linearly independent, but it does not span the space so is not a basis. For example, the list $\{(1,0,0),(0,1,0)\}$ is linearly independent, but spans only the *xy* plane.

A couple of examples of linear independence/dependence can be found here.

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