VECTOR SPACES: SPAN, LINEAR INDEPENDENCE AND BASIS

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References: edX online course MIT 8.05.1x Week 3.

Here, we investigate the ideas of the span of a vector space and see how this leads to the idea of linear independence of a set of vectors. I’ll summarize the main definitions and results here for future use; a more complete explanation together with some examples is given in Axler’s book, Chapter 2.

Span of a list of vectors. A list of vectors is just a subset of the vectors in a vector space, with the condition that the number of vectors in the subset is finite. The set of all linear combinations of the vectors \((v_1, \ldots, v_m)\) in a list is called the span of that list. Since a general linear combination has the form

\[ v = \sum_{i=1}^{m} a_i v_i \]  

where \(a_i \in \mathbb{F}\) (recall that the field \(\mathbb{F}\) is always taken to be either the real numbers \(\mathbb{R}\) or the complex numbers \(\mathbb{C}\)), the span of a list itself forms a vector space which is a subspace of the original vector space. One result we can show is

**Theorem 1.** The span of a list of vectors in a vector space \(V\) is the smallest subspace of \(V\) containing all the vectors in the list.

**Proof.** Let the list be \(L \equiv (v_1, \ldots, v_m)\). Then \(S \equiv \text{span}(v_1, \ldots, v_m)\) is a subspace since it contains the zero vector if all \(a_i\)s are zero in [1] and since it contains all linear combinations of the list, it is closed under addition and scalar multiplication.

The span \(S\) contains all \(v_j \in L\) (just set \(a_j = \delta_{ij}\) in [1]). Now if we look at a subspace of \(V\) that contains all the \(v_i\)s, it must also contain every vector in the span \(S\), since a subspace must be closed under addition and scalar multiplication. Thus \(S\) is the smallest subspace of \(V\) that contains all the vectors in \(L\). \(\square\)

If \(S \equiv \text{span}(v_1, \ldots, v_m) = V\), that is, the span of a list is the same as the original vector space, then we say that \((v_1, \ldots, v_m)\) spans \(V\). This leads to
the definition that a vector space is called finite-dimensional if it is spanned by some list of vectors. (Remember that all lists are finite in length!) A vector space that is not finite-dimensional is called (not surprisingly) infinite-dimensional.

**Linear independence.** Suppose a list \((v_1, \ldots, v_m) \in V\) and \(v\) is a vector such that \(v \in \text{span}(v_1, \ldots, v_m)\). This means that \(v\) is a linear combination of \((v_1, \ldots, v_m)\), so that [1] is true. However, using only the definitions above, there is no guarantee that there is only one choice for the scalars \(a_i\) that satisfies [1]. We might also have, for example

\[
v = \sum_{i=1}^{m} c_i v_i
\]

where \(c_i \neq a_i\). This means that we can write the zero vector as

\[
0 = \sum_{i=1}^{m} (a_i - c_i) v_i
\]

Now, if the only way we can satisfy this equation is to require that \(a_i = c_i\) for all \(i\), then we say that the list \((v_1, \ldots, v_m)\) is linearly independent. (For completeness, the empty list (containing no vectors) is also declared to be linearly independent.) By reversing the above argument, we see that if the list \((v_1, \ldots, v_m)\) is linearly independent, then there is only one set of scalars \(a_i\) such that [1] is satisfied. In other words, any vector \(v \in \text{span}(v_1, \ldots, v_m)\) has only one representation as a linear combination of the vectors in the list.

A list that is not linearly independent is, again not surprisingly, defined to be linearly dependent. This leads to the linear dependence lemma:

**Lemma 2.** Suppose \((v_1, \ldots, v_m)\) is a linearly dependent list in \(V\). Then there exists some \(j \in \{1, 2, \ldots, m\}\) such that

(a) \(v_j \in \text{span}(v_1, \ldots, v_{j-1})\);

(b) if \(v_j\) is removed from the list \((v_1, \ldots, v_m)\), the span of the remaining list, containing \(m - 1\) vectors, equals the span of the original list.

**Proof.** Because \((v_1, \ldots, v_m)\) is linearly dependent, we can write

\[
\sum_{i=1}^{m} a_i v_i = 0
\]

where not all of the \(a_i\)s are zero. Suppose \(j\) is the largest index where \(a_j \neq 0\). Then we can divide through by \(a_j\) to get

\[
v_j = -\frac{1}{a_j} \sum_{i=1}^{j-1} a_i v_i
\]
Thus \( v_j \) is a linear combination of other vectors in the list, which proves part (a). Part (b) follows from the fact that we can represent any vector \( u \in \text{span} (v_1, \ldots, v_m) \) as

\[
    u = \sum_{i=1}^{m} a_i v_i
\]

We can replace \( v_j \) in this sum by \( \Box \), so \( u \) can be written as a linear combination of all the vectors in the list \( (v_1, \ldots, v_m) \) except for \( v_j \). Thus (b) is true. \( \Box \)

We can use this lemma to prove the main result about linearly independent lists:

**Theorem 3.** In a finite-dimensional vector space \( V \), the length of every linearly independent list is less than or equal to the length of every list that spans \( V \).

**Proof.** Suppose the list \( A \equiv (u_1, \ldots, u_m) \) is linearly independent in \( V \), and suppose another list \( B \equiv (w_1, \ldots, w_n) \) spans \( V \). We want to prove that \( m \leq n \).

Since \( B \) already spans \( V \), if we add any other vector from \( V \) to the list \( B \), we will get a linearly dependent list, since this newly added vector can, by the definition of a span, be expressed a linear combination of the vectors in \( B \). In particular, if we add \( u_1 \) from the list \( A \) to \( B \), then the list \( (u_1, w_1, \ldots, w_n) \) is linearly dependent. By the linear independence lemma above, we can therefore remove one of the \( w_i \)s from \( B \) so that the remaining list still spans \( V \), and contains \( n \) vectors. For the sake of argument, let’s say we remove \( w_n \) (we can always order the \( w_i \)s in the list so that the element we remove is at the end). Then we’re left with the revised list \( B_1 = (u_1, w_1, \ldots, w_{n-1}) \).

We can repeat this process \( m \) times, each time adding the next element \( u_i \) from list \( A \) and removing the last \( w_i \). Because of the linear dependence lemma, we know that there must always be a \( w_i \) that can be removed each time we add a \( u_i \), so there must be at least as many \( w_i \)s as \( u_i \)s. In other words, \( m \leq n \) which is what we wanted to prove. \( \Box \)

This theorem can be used to show easily that any list of more than \( n \) vectors in \( n \)-dimensional space cannot be linearly independent, since we know that we can span \( n \)-dimensional space with \( n \) vectors (for example, the 3 coordinate axes in 3-d space). Conversely, since we can find a list of \( n \) vectors in \( n \)-dimensional space that is linearly independent, any list of fewer than \( n \) vectors cannot span \( n \)-dimensional space.
Basis of a finite-dimensional vector space. A basis of a finite-dimensional vector space is defined to be a list that is both linearly independent and spans the space. The dimension of the vector space is defined to be the length of a basis list. For example, in 3-d space, the list \{(1,0,0),(0,1,0),(0,0,1)\} is a basis, and since the length is 3, the dimension of the vector space is also 3. Any proper subset (that is, a subset with fewer than 3 members) of this basis is also linearly independent, but it does not span the space so is not a basis. For example, the list \{(1,0,0),(0,1,0)\} is linearly independent, but spans only the \(xy\) plane.

A couple of examples of linear independence/dependence can be found here.

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