LINEAR OPERATORS

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References: edX online course MIT 8.05.1x Week 3.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 3.

Having looked at some of the properties of a vector space, we can now look at *linear maps*. A linear map T is defined as a function that maps one vector space V into another (possibly the same) vector space W, written as

$$T : V \to W \tag{1}$$

The linear map T must satisfy the two properties

(1) **Additivity**: $T(u+v) = Tu + Tv$ for all $u, v \in V$.
(2) **Homogeneity**: $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$. As usual, the field $\mathbb{F}$ is either the set of real or complex numbers.

This definition of a linear map is general in the sense that the two vector spaces V and W can be any two vector spaces. In physics, it's more common to have $V = W$, and in such a case, the linear map T is called a *linear operator*.

With a couple of extra definitions, the set $\mathcal{L}(V)$ of all linear operators on V is itself a vector space, with the operators being the vectors. In order for this to be true, we need the following:

(1) **Zero operator**: A zero operator, written as just 0 (the same symbol now being used for three distinct objects: the scalar 0, the vector 0 and the operator 0; again the correct meaning is usually easy to deduce from the context) which has the property that the result of acting with 0 on any vector produces the 0 vector. That is $0v = 0$, where the 0 on the LHS is the zero operator and the 0 on the RHS is the zero vector.
(2) **Identity operator**: An identity operator I (sometimes written as 1) leaves any vector unchanged, so that $Iv = v$ for all $v \in V$.

With these definitions, $\mathcal{L}(V)$ is now a vector space, since it satisfies the distributive (additivity) and scalar multiplication (homogeneity) properties, contains an additive identity (the zero operator) and a multiplicative identity (the identity operator).

In addition, there is a natural definition of the multiplication of two linear operators S and T, written as ST. When a product operates on a vector $v \in V$, we just operate from right to left in succession, so that

$$(ST)v = S(Tv) \tag{2}$$

The product of two operators produces another operator also in $\mathcal{L}(V)$, since this product also satisfies additivity and homogeneity:

$$\begin{aligned}
(ST)(u+v) &= S(T(u+v)) && (3)\\
&= S(Tu+Tv) && (4)\\
&= STu+STv && (5)\\
(ST)(\lambda v) &= S(T(\lambda v)) && (6)\\
&= S(\lambda Tv) && (7)\\
&= \lambda S(Tv) && (8)\\
&= \lambda(ST)v && (9)
\end{aligned}$$

A very important property of operator multiplication is that it is not commutative. We've already seen many examples of this in our journey through quantum mechanics with operators such as position and momentum, angular momentum and so on. The non-commutativity is a fundamental mathematical property however, and can be seen in other examples that have nothing to do with quantum theory.

For example, consider the left shift operator L and right shift operator R, defined to act on the vector space consisting of infinite sequences of numbers. That is, our vector space V is such that

$$v = (x_1, x_2, x_3, \ldots) \tag{10}$$

where $x_i \in \mathbb{F}$. The shift operators have the following effects:

$$\begin{aligned}
Lv &= (x_2, x_3, \ldots) && (11)\\
Rv &= (0, x_1, x_2, x_3, \ldots) && (12)
\end{aligned}$$

The L operator removes the first element in the sequence, while the R operator inserts a 0 (number!) as the new first element in the sequence. Note that 0 is the only number we could insert into a sequence in order that R be a linear operator, since from additivity above, we must have $R0 = 0$. That is, if we start with $v = 0$ (the vector all of whose elements $x_i = 0$), then $R0$ must also give the zero vector.

The two products LR and RL produce different results:

$$LRv = L(0, x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots) = v \tag{13}$$
$$RLv = R(x_2, x_3, \ldots) = (0, x_2, x_3, \ldots) \neq v \tag{14}$$

The difference $[L,R] \equiv LR - RL$ is called the *commutator* of the two operators L and R. If we introduce the operator which projects out the first element in the sequence:

$$P_1 v \equiv P_1(x_1, x_2, x_3, \ldots) = (x_1, 0, 0, \ldots) \tag{15}$$
$$Iv - P_1 v = (x_1, x_2, x_3, \ldots) - (x_1, 0, 0, \ldots) \tag{16}$$
$$= (0, x_2, x_3, \ldots) \tag{17}$$

then we have

$$[L,R]v = Iv - (Iv - P_1 v) \tag{18}$$
$$= P_1 v \tag{19}$$

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