

## INVERSES OF LINEAR OPERATORS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: edX online course MIT 8.05.1x Week 3.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 3.

On a vector space  $V$ , a linear operator  $T$  has an inverse  $S$  if  $TSv = STv = v$  for all  $v \in V$ . Here, we're restricting the arguments given in Axler's section 3.D by assuming that all linear operators act from  $V$  back onto  $V$ . (Axler's arguments allow  $T$  to map vectors from  $V$  into another vector space  $W$ .) We can show first that  $S$  is unique:

**Theorem 1.** *If a linear operator  $T$  has an inverse, the inverse is unique.*

*Proof.* Suppose that there are two distinct inverses  $S_1$  and  $S_2$ . Then

$$S_1 = S_1 I = S_1 (TS_2) = (S_1 T) S_2 = I S_2 = S_2$$

We've made use of the fact that  $TS_2 = S_1 T = I$ , the identity operator, by the definition of the inverse, and we've also used the associativity property of a vector space to go from the third to the fourth term.  $\square$

Because the inverse is unique, we can refer to it by the notation  $T^{-1}$ .

There is an important general result about operator inverses:

**Theorem 2.** *A linear operator is invertible if and only if it is both injective and surjective.*

*Proof.* We first recall the definitions of injective and surjective. An injective operator is one-to-one, so that if  $Tv_1 = Tv_2$ , then  $v_1 = v_2$ . A surjective operator has the entire vector space  $V$  as its range.

As this is an 'if and only if' proof, we need to prove the theorem in both directions. First, assume that  $T^{-1}$  exists, so that the operator is invertible. Then for  $u, v \in V$  suppose that  $Tu = Tv$ . Applying the inverse, we get

$$T^{-1}Tu = u = T^{-1}Tv = v \tag{1}$$

so we must have  $u = v$ , making  $T$  injective.

Next, to prove that the existence of an inverse implies surjectivity, we note that we can write any vector  $v \in V$  as

$$v = T \left( T^{-1}v \right) \tag{2}$$

That is, there is a vector  $T^{-1}v$  which, when operated on by  $T$ , gives any vector  $v \in V$ . Thus every vector  $v \in V$  is in the range of  $T$ , making  $T$  surjective.

Now we need to prove that if  $T$  is both injective and surjective, it has an inverse. Let the operator  $S$  be defined by the property that for any vector  $w \in V$ ,  $v = Sw$  gives a vector  $v \in V$  such that  $Tv = w$ . Because  $T$  is injective, we know that  $v = Sw$  must be unique, since only one vector  $v$  can satisfy  $Tv = w$ . We also know that  $v$  must exist for every  $w \in V$  because  $T$  is surjective, and must produce every vector in  $V$  in its range. Since  $Tv = T(Sw) = (TS)w = w$ , we must have  $TS = I$ , so  $S$  is an inverse on the RHS side of  $T$ . To prove that  $ST = I$  as well, we have

$$T(STv) = (TS)Tv = ITv = Tv \quad (3)$$

Comparing first and last terms, we see that  $ST = I$ . Thus  $TS = ST = I$  and  $S = T^{-1}$  so  $T$  has an inverse.

[The full proof also requires that we show that  $S$  is linear; the details are done in Axler's theorem 3.56 if you're interested.]  $\square$

In the above proof, we did not need to assume anything about the dimension of the vector space  $V$ , so that the result is valid for both finite and infinite-dimensional vector spaces. For a finite vector space, the result can be made even stricter. The fundamental theorem of linear maps states that for a finite vector space  $V$  and linear operator  $T$

$$\dim V = \dim \text{null } T + \dim \text{range } T \quad (4)$$

If  $T$  is injective, then  $\dim \text{null } T = 0$ , since  $0$  is the only vector in the null space. Thus  $\dim \text{range } T = \dim V$  and  $T$  is surjective. Conversely, if we know that  $\dim \text{range } T = \dim V$ , then the theorem tells us that the null space has  $0$  dimensions. In other words, for a finite-dimensional vector space, injectivity implies surjectivity and vice versa. Thus, in this case, if we know that  $T$  is either injective or surjective, we know it has an inverse.