INVERSES OF LINEAR OPERATORS

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References: edX online course MIT 8.05.1x Week 3.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 3.

On a vector space V, a linear operator T has an inverse S if $TSv = STv = v$ for all $v \in V$. Here, we're restricting the arguments given in Axler's section 3.D by assuming that all linear operators act from V back onto V. (Axler's arguments allow T to map vectors from V into another vector space W.) We can show first that S is unique:

Theorem 1. *If a linear operator T has an inverse, the inverse is unique.*

Proof. Suppose that there are two distinct inverses S_1 and S_2. Then

$$S_1 = S_1 I = S_1 (TS_2) = (S_1 T) S_2 = IS_2 = S_2$$

We've made use of the fact that $TS_2 = S_1 T = I$, the identity operator, by the definition of the inverse, and we've also used the associativity property of a vector space to go from the third to the fourth term. □

Because the inverse is unique, we can refer to it by the notation T^{-1}.

There is an important general result about operator inverses:

Theorem 2. *A linear operator is invertible if and only if it is both injective and surjective.*

Proof. We first recall the definitions of injective and surjective. An injective operator is one-to-one, so that if $Tv_1 = Tv_2$, then $v_1 = v_2$. A surjective operator has the entire vector space V as its range.

As this is an 'if and only if' proof, we need to prove the theorem in both directions. First, assume that T^{-1} exists, so that the operator is invertible. Then for $u, v \in V$ suppose that $Tu = Tv$. Applying the inverse, we get

$$T^{-1}Tu = u = T^{-1}Tv = v \tag{1}$$

so we must have $u = v$, making T injective.

Next, to prove that the existence of an inverse implies surjectivity, we note that we can write any vector $v \in V$ as

$$v = T\left(T^{-1}v\right) \tag{2}$$

That is, there is a vector $T^{-1}v$ which, when operated on by T, gives any vector $v \in V$. Thus every vector $v \in V$ is in the range of T, making T surjective.

Now we need to prove that if T is both injective and surjective, it has an inverse. Let the operator S be defined by the property that for any vector $w \in V$, $v = Sw$ gives a vector $v \in V$ such that $Tv = w$. Because T is injective, we know that $v = Sw$ must be unique, since only one vector v can satisfy $Tv = w$. We also know that v must exist for every $w \in V$ because T is surjective, and must produce every vector in V in its range. Since $Tv = T(Sw) = (TS)w = w$, we must have $TS = I$, so S is an inverse on the RHS side of T. To prove that $ST = I$ as well, we have

$$T(STv) = (TS)Tv = ITv = Tv \tag{3}$$

Comparing first and last terms, we see that $ST = I$. Thus $TS = ST = I$ and $S = T^{-1}$ so T has an inverse.

[The full proof also requires that we show that S is linear; the details are done in Axler's theorem 3.56 if you're interested.] □

In the above proof, we did not need to assume anything about the dimension of the vector space V, so that the result is valid for both finite and infinite-dimensional vector spaces. For a finite vector space, the result can be made even stricter. The fundamental theorem of linear maps states that for a finite vector space V and linear operator T

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \tag{4}$$

If T is injective, then $\dim \operatorname{null} T = 0$, since 0 is the only vector in the null space. Thus $\dim \operatorname{range} T = \dim V$ and T is surjective. Conversely, if we know that $\dim \operatorname{range} T = \dim V$, then the theorem tells us that the null space has 0 dimensions. In other words, for a finite-dimensional vector space, injectivity implies surjectivity and vice versa. Thus, in this case, if we know that T is either injective or surjective, we know it has an inverse.