MATRIX REPRESENTATION OF LINEAR OPERATORS: CHANGE OF BASIS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: edX online course MIT 8.05.1x Week 3.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 3.

We've seen that the matrix representation of a linear operator depends on the basis we've chosen within a vector space V. We now look at how the matrix representation changes if we change the basis. In what follows, we'll consider two sets of basis vectors $\{v\}$ and $\{u\}$ and two operators A and B. Operator A transforms the basis $\{v\}$ into the basis $\{u\}$, while B does the reverse. That is

$$Av_i = u_i \tag{1}$$
$$Bu_i = v_i \tag{2}$$

for all $i = 1,\ldots,n$. From this definition, we can see that $A = B^{-1}$ and $B = A^{-1}$, since

$$u_i = Av_i = ABu_i \tag{3}$$
$$v_i = Bu_i = BAv_i \tag{4}$$

Theorem 1. *An operator (like A or B above) that transforms one set of basis vectors into another has the same matrix representation in both bases.*

Proof. In matrix form, we have (remember we're using the summation convention on repeated indices):

$$Av_i = A_{ji}(\{v\})\,v_j \tag{5}$$
$$Au_i = A_{ji}(\{u\})\,u_j \tag{6}$$

Note that the matrix elements depend on *different* bases in the two equations.

We can now operate with A again, using 1, to get

$$\begin{aligned} Au_i &= A(Av_i) && (7)\\ &= A(A_{ji}(\{v\})v_j) && (8)\\ &= A_{ji}(\{v\})Av_j && (9)\\ &= A_{ji}(\{v\})u_j && (10)\end{aligned}$$

Comparing the last line with 6, we see that

$$A_{ji}(\{v\}) = A_{ji}(\{u\})$$

Since the matrix elements are just numbers, this means that the elements in the two matrices $A_{ji}(\{v\})$ and $A_{ji}(\{u\})$ are the same.

We could do the same analysis using the B operator with the same result:

$$B_{ji}(\{v\}) = B_{ji}(\{u\}) \tag{11}$$

$\square$

We can now turn to the matrix representations of a general operator T in two different bases. In this case, T can perform any linear transformation, so it doesn't necessarily transform one set of basis vectors into another set of basis vectors. Consider first the case where T operates on each set of basis vectors given above:

$$\begin{aligned} Tv_i &= T_{ji}(\{v\})v_j && (12)\\ Tu_i &= T_{ji}(\{u\})u_j && (13)\end{aligned}$$

Unless T is an operator like A or B above, in general $T_{ji}(\{v\}) \neq T_{ji}(\{u\})$. We can see how these two matrices are related by using operators A and B above to write

$$\begin{aligned} Tu_i &= T(A_{ji}v_j) && (14)\\ &= A_{ji}Tv_j && (15)\\ &= A_{ji}T_{kj}(\{v\})v_k && (16)\\ &= A_{ji}T_{kj}(\{v\})Bu_k && (17)\\ &= A_{ji}T_{kj}(\{v\})A^{-1}u_k && (18)\\ &= A_{ji}T_{kj}(\{v\})A^{-1}_{pk}u_p && (19)\\ &= \left[A^{-1}_{pk}T_{kj}(\{v\})A_{ji}\right]u_p && (20)\\ &= T_{pi}(\{u\})u_p && (21)\end{aligned}$$

We don't need to specify the basis for the A or B matrices since the matrices are the same in both bases as we just saw above. The last line is just the expansion of Tu_i in terms of the $\{u\}$ basis. In the penultimate line, we see that the quantity in square brackets is the product of 3 matrices:

$$A_{pk}^{-1}T_{kj}(\{v\})A_{ji} = \left[A^{-1}T(\{v\})A\right]_{pi} \tag{22}$$

The required transformation is therefore

$$T(\{u\}) = A^{-1}T(\{v\})A \tag{23}$$

where $u_i = Av_i$.

As a check, note that if $T = A$ or $T = B = A^{-1}$, we reclaim the result in the theorem above, namely that $A(\{u\}) = A(\{v\})$ and $B(\{u\}) = B(\{v\})$.

Trace and determinant. The trace of a matrix is the sum of its diagonal elements, written as tr T. A useful property of the trace is that

$$\text{tr }(AB) = \text{tr }(BA) \tag{24}$$

We can prove this by looking at the components. If $C = AB$ then

$$C_{ij} = A_{ik}B_{kj} \tag{25}$$

The trace of C is the sum of its diagonal elements, written as C_{ii}, so

$$\begin{aligned}
\text{tr } C &= \text{tr }(AB) && (26)\\
&= A_{ik}B_{ki} && (27)\\
&= B_{ki}A_{ik} && (28)\\
&= [BA]_{kk} && (29)\\
&= \text{tr }(BA) && (30)
\end{aligned}$$

From this we can generalize to the case of the trace of a product of any number of matrices and obtain the cyclic rule:

$$\text{tr}(A_1A_2\ldots A_n) = \text{tr}(A_nA_1A_2\ldots A_{n-1}) \tag{31}$$

Going back to 23, we have

$$\begin{aligned}
\text{tr } T(\{u\}) &= \text{tr}\left(A^{-1}T(\{v\})A\right) && (32)\\
&= \text{tr}\left(AA^{-1}T(\{v\})\right) && (33)\\
&= \text{tr } T(\{v\}) && (34)
\end{aligned}$$

Thus the trace of any linear operator is invariant under a change of basis.

For the determinant, we have the results that the determinant of a product of matrices is equal to the product of the determinants, and the determinant of a matrix inverse is the reciprocal of the determinant of the original matrix. Therefore

$$\det\left(T\left(\{u\}\right)\right) = \det\left(A^{-1}T\left(\{v\}\right)A\right) \tag{35}$$

$$= \frac{\det A}{\det A}\det T\left(\{v\}\right) \tag{36}$$

$$= \det T\left(\{v\}\right) \tag{37}$$

Thus the determinant is also invariant under a change of basis.

PINGBACKS

Pingback: Projection operators
Pingback: Unitary operators
Pingback: Diagonalization of matrices
Pingback: Unitary operators: active and passive transformations of an operator
Pingback: Pauli matrices: properties