INNER PRODUCTS AND HILBERT SPACES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: edX online course MIT 8.05.1x Week 4.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 6.

An inner product defined on a vector space V is a function that maps each ordered pair of vectors (u,v) of V to a number denoted by $\langle u,v\rangle \in \mathbb{F}$. The inner product satisfies the following axioms:

(1) positivity: $\langle v,v\rangle \geq 0$ for all $v \in V$.
(2) definiteness: $\langle v,v\rangle = 0$ if and only if $v = 0$ (the zero vector).
(3) additivity in the second slot: $\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$.
(4) homogeneity in the second slot: $\langle u,\lambda v\rangle = \lambda\langle u,v\rangle$ where $\lambda \in \mathbb{F}$.
(5) conjugate symmetry: $\langle u,v\rangle = \langle v,u\rangle^*$.

[Axler requires additivity and homegeneity in the first slot rather than the second, but Zwiebach uses the conditions above, which are more usual for physics.]

The *norm* $|v|$ of a vector v is defined as

$$|v|^2 \equiv \langle v,v\rangle \tag{1}$$

All the above applies to both real and complex vector spaces, although it should be noted that in the case of conjugate symmetry in a real space, $\langle v,u\rangle^* = \langle v,u\rangle$ so in that case, the condition reduces to $\langle u,v\rangle = \langle v,u\rangle$.

Conditions 3 and 4 apply to the first slot as well, though with a slight difference for complex vector spaces. [These properties can actually be derived from the 5 axioms above, so aren't listed as separate axioms]:

$$\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle \tag{2}$$

$$\langle \lambda u,v\rangle = \lambda^*\langle u,v\rangle \tag{3}$$

Two vectors are *orthogonal* if $\langle u,v\rangle = \langle v,u\rangle = 0$. By this definition, the zero vector is orthogonal to all vectors, including itself.

The inner product is *non-degenerate*, meaning that any vector that is orthogonal to all vectors must be zero.

An *orthogonal decomposition* is defined for a vector u as follows: Suppose $u,v \in V$ and $v \neq 0$. We can write u as

$$u = \frac{\langle u,v\rangle}{|v|^2}v + u - \frac{\langle u,v\rangle}{|v|^2}v \tag{4}$$
$$\equiv cv + w \tag{5}$$

where

$$c = \frac{\langle u,v\rangle}{|v|^2} \in \mathbb{F} \tag{6}$$
$$w = u - \frac{\langle u,v\rangle}{|v|^2}v \tag{7}$$

From the definition, $\langle w,v\rangle = 0$ so we've decomposed u into a component cv 'parallel' to v and another component w orthogonal to v.

There are a couple of important theorems for which we'll run through the proofs:

Theorem 1. *Pythagorean theorem. If u and v are orthogonal then*

$$|u+v|^2 = |u|^2 + |v|^2 \tag{8}$$

Proof. From the definition of the norm

$$|u+v|^2 = \langle u+v, u+v\rangle \tag{9}$$
$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \tag{10}$$
$$= \langle u,u\rangle + \langle v,v\rangle \tag{11}$$
$$= |u|^2 + |v|^2 \tag{12}$$

where in the third line we used the orthogonality condition $\langle u,v\rangle = \langle v,u\rangle = 0$. □

Theorem 2. *Schwarz (or Cauchy-Schwarz) inequality. For all vectors $u,v \in V$*

$$|\langle u,v\rangle| \le |u|\,|v| \tag{13}$$

Proof. The inequality is obviously true (as an equality) if $v = 0$, so we need to prove it for $v \neq 0$. In that case we can form an orthogonal decomposition of u:

$$u = \frac{\langle u,v\rangle}{|v|^2}v + w \tag{14}$$

Since v and w are orthogonal, we can apply the Pythagorean theorem:

$$
\begin{aligned}
|u|^2 &= \left|\frac{\langle u,v\rangle}{|v|^2}v\right|^2 + |w|^2 && (15)\\
&= \frac{|\langle u,v\rangle|^2}{|v|^4}|v|^2 + |w|^2 && (16)\\
&= \frac{|\langle u,v\rangle|^2}{|v|^2} + |w|^2 && (17)\\
&\geq \frac{|\langle u,v\rangle|^2}{|v|^2} && (18)
\end{aligned}
$$

Rearranging the last line and taking the positive square root (since norms are always non-negative) we have

$$|\langle u,v\rangle| \leq |u|\,|v| \tag{19}$$

□

Theorem 3. *Triangle inequality. For all $u,v \in V$*

$$|u+v| \leq |u| + |v| \tag{20}$$

Proof. As with the Schwarz inequality, we start with the square of the norm:

$$
\begin{aligned}
|u+v|^2 &= \langle u+v, u+v\rangle && (21)\\
&= \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle && (22)\\
&= \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle u,v\rangle^* && (23)\\
&= |u|^2 + |v|^2 + 2\Re\langle u,v\rangle && (24)\\
&\leq |u|^2 + |v|^2 + 2|\langle u,v\rangle| && (25)
\end{aligned}
$$

The last line follows because $\langle u,v\rangle$ is a complex number, so

$$|\langle u,v\rangle| = \sqrt{(\Re\langle u,v\rangle)^2 + (\Im\langle u,v\rangle)^2} \geq \Re\langle u,v\rangle \tag{26}$$

We can now apply the Schwarz inequality to the last term in the last line to get

$$
\begin{aligned}
|u+v|^2 &\leq |u|^2 + |v|^2 + 2|u|\,|v| && (27)\\
&= (|u| + |v|)^2 && (28)
\end{aligned}
$$

Taking the positive square root, we get

$$|u+v| \leq |u| + |v| \tag{29}$$

$\square$

A finite-dimensional complex vector space with an inner product is a *Hilbert space.* An infinite-dimensional complex vector space with an inner product is also a Hilbert space if a completeness property holds. This property is a technical property which is always satisfied in quantum mechanics, so we can assume that any infinite-dimensional complex vector spaces we encounter in quantum theory are Hilbert spaces.

PINGBACKS

Pingback: Orthonormal basis and orthogonal complement
Pingback: Projection operators
Pingback: Linear functionals and adjoint operators
Pingback: Triangle inequality as an equality
Pingback: Non-denumerable basis: position and momentum states
Pingback: Direct product of two vector spaces