ORTHONORMAL BASIS AND ORTHOGONAL COMPLEMENT

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References: edX online course MIT 8.05.1x Week 4.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 6.

Once we have defined an inner product defined on a vector space V, we can create an orthonormal basis for V. A list of vectors $(e_1, e_2, \ldots, e_n)$ is orthonormal if

$$\left\langle e_i, e_j \right\rangle = \delta_{ij} \tag{0.1}$$

That is, any pair of vectors is orthogonal, and all the vectors have norm 1. In 3-d space, the unit vectors along the three axes form an orthonormal list.

Given an orthonormal list, we can construct a vector from the vectors in that list by

$$v = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n \tag{0.2}$$

$$= \sum_{i=1}^{n} a_i e_i \tag{0.3}$$

for $a_i \in \mathbb{F}$. The norm of v has a simple form:

$$\left\langle v, v \right\rangle^2 = \left\langle \sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} a_i e_i \right\rangle \tag{0.4}$$

$$= \sum_{i=1}^{n} \left\langle a_i e_i, a_i e_i \right\rangle + \text{zero terms} \tag{0.5}$$

$$= \sum_{i=1}^{n} |a_i|^2 \tag{0.6}$$

The 'zero terms' in the second line are terms involving $\left\langle a_i e_i, a_j e_j \right\rangle$ for $i \neq j$ which are all zero because of 0.1.

This result shows that an orthonormal list of vectors is linearly independent, since if we form the linear combination

$$v = a_1e_1 + a_2e_2 + \ldots + a_ne_n = 0 \tag{0.7}$$

then $\langle v, v\rangle = 0$ so from 0.6 we must have all $a_i = 0$, which means the list is linearly independent.

If we have an orthonormal list $(e_1, e_2, \ldots, e_n)$ that is also a basis for V, then any vector $v \in V$ can be written as

$$v = a_1e_1 + a_2e_2 + \ldots + a_ne_n \tag{0.8}$$

The coefficients a_i can be found by taking the inner product $\langle e_i, v\rangle = a_i$ (using 0.1), so we have

$$v = \sum_{i=1}^{n} \langle e_i, v\rangle e_i \tag{0.9}$$

For example, in 3-d space, the 3 unit vectors along the x, y, z axes form an orthonormal basis for the space. However, the unit vectors along the x, y axes form an orthonormal list, but this is not a basis for 3-d space since no vector with a z component can be written as a linear combination of these two vectors.

If we have any basis (not necessarily orthonormal), we can form an orthonormal basis using the Gram-Schmidt orthogonalization procedure. We've already met this in the context of quantum mechanics, and the derivation for a general finite vector space is much the same, so I'll just quote the result. The procedure is iterative and follows these steps:

The first vector e_1 in the orthonormal basis is defined by

$$e_1 = \frac{v_1}{|v_1|} \tag{0.10}$$

where v_1 is the first vector (well, any vector, really) in the non-orthonormal basis.

Given vector e_{j-1} in the orthonormal basis, we can form e_j from the formula

$$e_j = \frac{v_j - \sum_{i=1}^{j-1} \langle e_i, v_j\rangle e_i}{\left|v_j - \sum_{i=1}^{j-1} \langle e_i, v_j\rangle e_i\right|} \tag{0.11}$$

e_j clearly has norm 1, and we can check that $\langle e_i, e_j\rangle = \delta_{ij}$ by direct calculation. Note that although we've indexed the vectors v_i in the original basis, we can take them in any order when calculating the orthonormal basis via the Gram-Schmidt procedure.

Orthogonal complement. Suppose we have a subset U (not necessarily a subspace) of V. Then we can define the orthogonal complement $U^{\perp}$ of U as the set of all vectors that are orthogonal to all vectors $u \in U$. More formally:

$$U^{\perp} \equiv \{v \in V \,|\, \langle v,u\rangle = 0 \text{ for all } u \in U\} \tag{0.12}$$

A useful general theorem is as follows.

Theorem 1. *If U is a subspace of V, then $V = U \oplus U^{\perp}$. (Recall the direct sum.)*

Proof. Given an orthonormal basis of U: $(e_1, e_2, \ldots, e_n)$, we can write any $v \in V$ as the sum

$$v = \underbrace{\sum_{i=1}^{n} \langle e_i, v\rangle\, e_i}_{\in U} + \underbrace{v - \sum_{i=1}^{n} \langle e_i, v\rangle\, e_i}_{\in U^{\perp}} \tag{0.13}$$

On the RHS, we've just added and subtracted the same term from v. Since the first term is a linear combination of the basis vectors of U, the overall sum is a vector in U. To see that the second term is in $U^{\perp}$, take the inner product with any of the basis vectors e_k:

$$\left\langle e_k, v - \sum_{i=1}^{n} \langle e_i, v\rangle\, e_i \right\rangle = \langle e_k, v\rangle - \left\langle e_k, \sum_{i=1}^{n} \langle e_i, v\rangle\, e_i \right\rangle \tag{0.14}$$

$$= \langle e_k, v\rangle - \sum_{i=1}^{n} \langle e_i, v\rangle\, \delta_{ik} \tag{0.15}$$

$$= \langle e_k, v\rangle - \langle e_k, v\rangle \tag{0.16}$$

$$= 0 \tag{0.17}$$

Finally, since the two vector spaces in a direct sum can have only the zero vector in their intersection, we need to show that $U \cap U^{\perp} = \{0\}$. However if a vector v is in both U and $U^{\perp}$ then it must be orthogonal to itself, so $\langle v, v\rangle = 0$ which implies $v = 0$. $\square$

Thus any vector space V can be decomposed into two orthogonal subspaces (assuming that V has any subspaces other than $\{0\}$).

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