

## ORTHONORMAL BASIS AND ORTHOGONAL COMPLEMENT

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References: edX online course MIT 8.05.1x Week 4.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 6.

Once we have defined an inner product defined on a vector space  $V$ , we can create an orthonormal basis for  $V$ . A list of vectors  $(e_1, e_2, \dots, e_n)$  is orthonormal if

$$(1) \quad \langle e_i, e_j \rangle = \delta_{ij}$$

That is, any pair of vectors is orthogonal, and all the vectors have norm 1. In 3-d space, the unit vectors along the three axes form an orthonormal list.

Given an orthonormal list, we can construct a vector from the vectors in that list by

$$(2) \quad v = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$$

$$(3) \quad = \sum_{i=1}^n a_i e_i$$

for  $a_i \in \mathbb{F}$ . The norm of  $v$  has a simple form:

$$(4) \quad \langle v, v \rangle^2 = \left\langle \sum_{i=1}^n a_i e_i, \sum_{i=1}^n a_i e_i \right\rangle$$

$$(5) \quad = \sum_{i=1}^n \langle a_i e_i, a_i e_i \rangle + \text{zero terms}$$

$$(6) \quad = \sum_{i=1}^n |a_i|^2$$

The 'zero terms' in the second line are terms involving  $\langle a_i e_i, a_j e_j \rangle$  for  $i \neq j$  which are all zero because of 1.

This result shows that an orthonormal list of vectors is linearly independent, since if we form the linear combination

$$(7) \quad v = a_1 e_1 + a_2 e_2 + \dots + a_n e_n = 0$$

then  $\langle v, v \rangle = 0$  so from 6 we must have all  $a_i = 0$ , which means the list is linearly independent.

If we have an orthonormal list  $(e_1, e_2, \dots, e_n)$  that is also a basis for  $V$ , then any vector  $v \in V$  can be written as

$$(8) \quad v = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$$

The coefficients  $a_i$  can be found by taking the inner product  $\langle e_i, v \rangle = a_i$  (using 1), so we have

$$(9) \quad v = \sum_{i=1}^n \langle e_i, v \rangle e_i$$

For example, in 3-d space, the 3 unit vectors along the  $x, y, z$  axes form an orthonormal basis for the space. However, the unit vectors along the  $x, y$  axes form an orthonormal list, but this is not a basis for 3-d space since no vector with a  $z$  component can be written as a linear combination of these two vectors.

If we have any basis (not necessarily orthonormal), we can form an orthonormal basis using the Gram-Schmidt orthogonalization procedure. We've already met this in the context of quantum mechanics, and the derivation for a general finite vector space is much the same, so I'll just quote the result. The procedure is iterative and follows these steps:

The first vector  $e_1$  in the orthonormal basis is defined by

$$(10) \quad e_1 = \frac{v_1}{|v_1|}$$

where  $v_1$  is the first vector (well, any vector, really) in the non-orthonormal basis.

Given vector  $e_{j-1}$  in the orthonormal basis, we can form  $e_j$  from the formula

$$(11) \quad e_j = \frac{v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|}$$

$e_j$  clearly has norm 1, and we can check that  $\langle e_i, e_j \rangle = \delta_{ij}$  by direct calculation. Note that although we've indexed the vectors  $v_i$  in the original basis, we can take them in any order when calculating the orthonormal basis via the Gram-Schmidt procedure.

**Orthogonal complement.** Suppose we have a subset  $U$  (not necessarily a subspace) of  $V$ . Then we can define the orthogonal complement  $U^\perp$  of  $U$  as the set of all vectors that are orthogonal to all vectors  $u \in U$ . More formally:

$$(12) \quad U^\perp \equiv \{v \in V \mid \langle v, u \rangle = 0 \text{ for all } u \in U\}$$

A useful general theorem is as follows.

**Theorem 1.** *If  $U$  is a subspace of  $V$ , then  $V = U \oplus U^\perp$ . (Recall the direct sum.)*

*Proof.* Given an orthonormal basis of  $U$ :  $(e_1, e_2, \dots, e_n)$ , we can write any  $v \in V$  as the sum

$$(13) \quad v = \underbrace{\sum_{i=1}^n \langle e_i, v \rangle e_i}_{\in U} + v - \underbrace{\sum_{i=1}^n \langle e_i, v \rangle e_i}_{\in U^\perp}$$

On the RHS, we've just added and subtracted the same term from  $v$ . Since the first term is a linear combination of the basis vectors of  $U$ , the overall sum is a vector in  $U$ . To see that the second term is in  $U^\perp$ , take the inner product with any of the basis vectors  $e_k$ :

$$(14) \quad \left\langle e_k, v - \sum_{i=1}^n \langle e_i, v \rangle e_i \right\rangle = \langle e_k, v \rangle - \left\langle e_k, \sum_{i=1}^n \langle e_i, v \rangle e_i \right\rangle$$

$$(15) \quad = \langle e_k, v \rangle - \sum_{i=1}^n \langle e_i, v \rangle \delta_{ik}$$

$$(16) \quad = \langle e_k, v \rangle - \langle e_k, v \rangle$$

$$(17) \quad = 0$$

Finally, since the two vector spaces in a direct sum can have only the zero vector in their intersection, we need to show that  $U \cap U^\perp = \{0\}$ . However if a vector  $v$  is in both  $U$  and  $U^\perp$  then it must be orthogonal to itself, so  $\langle v, v \rangle = 0$  which implies  $v = 0$ .  $\square$

Thus any vector space  $V$  can be decomposed into two orthogonal subspaces (assuming that  $V$  has any subspaces other than  $\{0\}$ ).

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