PROJECTION OPERATORS

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References: edX online course MIT 8.05.1x Week 4.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 6.

Continuing from our examination of orthonormal bases and the orthogonal complement in a vector space V, we can now look at the orthogonal projection, sometimes known in physics as a projection operator.

Suppose we have defined a subspace U of V and its orthogonal complement $U^{\perp}$, so that $V = U \oplus U^{\perp}$. We can define a linear operator P_U called the orthogonal projection operator. It has the property that, given any vector $v \in V$, it 'projects' out the component of v that lies in U. That is, if we write

$$v = u + w \tag{1}$$

where $u \in U$ and $w \in U^{\perp}$, then

$$P_U v = u \tag{2}$$

An example of a projection operator is an operator in 3-d space that projects a vector onto the xy plane. Then the xy plane is the subspace U and the z axis is the orthogonal complement $U^{\perp}$.

From the definition of P_U we can list a few properties:

(1) P_U is not surjective, that is, its range is smaller than the entire space V.
(2) P_U is not injective, since it maps all vectors $u + w$ to u, for all $w \in U^{\perp}$. Thus it is a many-to-one mapping.
(3) P_U is not invertible, since it is not injective.
(4) Its null space is null $P_U = U^{\perp}$.
(5) Once P_U is applied to any vector v, all subsequent applications of P_U have no effect. That is, once you've projected out the component of v that lies in U, all further projections into U just give the same result. In other words $P_U^n = P_U$ for all integers $n > 0$.
(6) $|P_U v| \le |v|$. This follows from the Pythagorean theorem, since u and w are orthogonal, so $|v|^2 = |u|^2 + |w|^2 \ge |u|^2 = |P_U v|^2$. Geometrically, a projection operator cannot increase the 'length' (norm) of a

vector. This property relies on the fact that the projection is an orthogonal projection. Other projections can increase the length of a vector (think of the shadow cast by a stick; if the surface onto which the shadow falls is nearly parallel to the direction of the incoming light, the shadow is much longer than the stick).

An explicit form for $P_U v$ can be obtained from the decomposition we had earlier

$$v = \underbrace{\sum_{i=1}^{n} \langle e_i, v\rangle e_i}_{\in U} + \underbrace{v - \sum_{i=1}^{n} \langle e_i, v\rangle e_i}_{\in U^\perp} \tag{3}$$

From this,

$$P_U v = \sum_{i=1}^{n} \langle e_i, v\rangle e_i \tag{4}$$

From the definition, it seems reasonable that a vector space V can be decomposed into a direct sum of range P_U and null P_U. We can in fact prove this.

Theorem 1. *P is an orthogonal projection within the vector space V if*

$$V = \text{null } P \oplus \text{range } P \tag{5}$$

Proof. We can take the subspace $U = \text{range } P$. From our earlier theorem, we know that $V = U \oplus U^\perp$, so we need to show that $U^\perp = \text{null } P$. Since $Pw = 0$ for any $w \in U^\perp$, then null $P \subset U^\perp$, but are there vectors in $U^\perp$ that are not in null P? Suppose there is such a vector $x \in U^\perp$ such that $Px \neq 0$. For such a vector, we can decompose it into $x = x' + x''$ where $x' \in \text{null } P$ and $x'' \in \text{range } P$, with $x'' \neq 0$ (since if $x'' = 0$, then x would be in null P, contrary to our assumption).

As $x \in U^\perp$, $\langle x, u\rangle = 0$ for all $u \in U = \text{range } P$. Therefore $\langle x, u\rangle = \langle x' + x'', u\rangle = \langle x', u\rangle + \langle x'', u\rangle = 0$. Since $x' \in \text{null } P$, $\langle x', u\rangle = 0$ (as $x' \in U^\perp$). Therefore we must have $\langle x'', u\rangle = 0$, implying that $x'' \in U^\perp$ also. Thus $x'' \in U$ and $x'' \in U^\perp$, but the only vector that can be in both a subspace and its orthogonal complement is 0, so $x'' = 0$, which contradicts our assumption above. □

From property 5 above, we must have $P_U^2 = P_U$, which implies that the eigenvalues of P_U are 0 and 1. The eigenvectors belong to either the subspace U (for eigenvalue 1) or to the orthogonal complement $U^\perp$ (for eigenvalue 0).

The orthonormal basis of a vector space V can be divided into two separate lists of vectors, with one list $(e_1,\ldots,e_m)$ spanning the subspace U and the other list $(f_1,\ldots,f_k)$ spanning $U^{\perp}$. A matrix representation of P_U can be obtained by considering the action of P_U on each of the basis vectors from the two subspaces. We have

$$P_U e_i = e_i \tag{6}$$

$$P_U f_i = 0 \tag{7}$$

In general, the matrix representation of an operator T is defined in terms of its action on the basis vectors v_i by

$$v'_j = \sum_{i=1}^{n} T_{ij} v_i \tag{8}$$

For a projection operator, we can see that this means that for the m basis vectors $(e_1,\ldots,e_m)$ we must have $P_{ij} = \delta_{ij}$ for all $i,j = 1,\ldots,m$, while for the k basis vectors $(f_1,\ldots,f_k)$ we must have $P_{ij} = 0$ for all $i,j = 1,\ldots,k$. If we list the basis vectors in the order $(e_1,\ldots,e_m,f_1,\ldots,f_k)$, then P_U is a $(m+k)\times(m+k)$ diagonal matrix with the diagonal elements in the top m rows equal to 1, and all other elements equal to zero.

In this basis, we see that $\det P_U = 0$ (because there is at least one zero element on the diagonal) and $\mathrm{tr}\, P_U = m$, which is the dimension of the subspace U. As the trace and determinant are invariant under a change of basis, these properties apply to any basis.

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