# PROJECTION OPERATORS

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References: edX online course MIT 8.05.1x Week 4.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 6.

Continuing from our examination of orthonormal bases and the orthogonal complement in a vector space $V$, we can now look at the orthogonal projection, sometimes known in physics as a projection operator.

Suppose we have defined a subspace $U$ of $V$ and its orthogonal complement $U^{\perp}$, so that $V = U \oplus U^{\perp}$. We can define a linear operator $P_U$ called the orthogonal projection operator. It has the property that, given any vector $v \in V$, it 'projects' out the component of $v$ that lies in $U$. That is, if we write

$$v = u + w \tag{1}$$

where $u \in U$ and $w \in U^{\perp}$, then

$$P_U v = u \tag{2}$$

An example of a projection operator is an operator in 3-d space that projects a vector onto the $xy$ plane. Then the $xy$ plane is the subspace $U$ and the $z$ axis is the orthogonal complement $U^{\perp}$.

From the definition of $P_U$ we can list a few properties:

(1) $P_U$ is not surjective, that is, its range is smaller than the entire space $V$.
(2) $P_U$ is not injective, since it maps all vectors $u + w$ to $u$, for all $w \in U^{\perp}$. Thus it is a many-to-one mapping.
(3) $P_U$ is not invertible, since it is not injective.
(4) Its null space is null $P_U = U^{\perp}$.
(5) Once $P_U$ is applied to any vector $v$, all subsequent applications of $P_U$ have no effect. That is, once you've projected out the component of $v$ that lies in $U$, all further projections into $U$ just give the same result. In other words $P_U^n = P_U$ for all integers $n > 0$.
(6) $|P_U v| \le |v|$. This follows from the Pythagorean theorem, since $u$ and $w$ are orthogonal, so $|v|^2 = |u|^2 + |w|^2 \ge |u|^2 = |P_U v|^2$. Geometrically, a projection operator cannot increase the 'length' (norm) of a vector. This property relies on the fact that the projection is an orthogonal projection. Other projections can increase the length of a

vector (think of the shadow cast by a stick; if the surface onto which the shadow falls is nearly parallel to the direction of the incoming light, the shadow is much longer than the stick).

An explicit form for $P_U v$ can be obtained from the decomposition we had earlier

$$v = \underbrace{\sum_{i=1}^{n} \langle e_i, v \rangle e_i}_{\in U} + \underbrace{v - \sum_{i=1}^{n} \langle e_i, v \rangle e_i}_{\in U^\perp} \tag{3}$$

From this,

$$P_U v = \sum_{i=1}^{n} \langle e_i, v \rangle e_i \tag{4}$$

From the definition, it seems reasonable that a vector space $V$ can be decomposed into a direct sum of range $P_U$ and null $P_U$. We can in fact prove this.

**Theorem 1.** *$P$ is an orthogonal projection within the vector space $V$ if*

$$V = \textit{null } P \oplus \textit{range } P \tag{5}$$

*Proof.* We can take the subspace $U = \text{range } P$. From our earlier theorem, we know that $V = U \oplus U^\perp$, so we need to show that $U^\perp = \text{null } P$. Since $Pw = 0$ for any $w \in U^\perp$, then null $P \subset U^\perp$, but are there vectors in $U^\perp$ that are not in null $P$? Suppose there is such a vector $x \in U^\perp$ such that $Px \neq 0$. For such a vector, we can decompose it into $x = x' + x''$ where $x' \in \text{null } P$ and $x'' \in \text{range } P$, with $x'' \neq 0$ (since if $x'' = 0$, then $x$ would be in null $P$, contrary to our assumption).

As $x \in U^\perp$, $\langle x, u \rangle = 0$ for all $u \in U = \text{range } P$. Therefore $\langle x, u \rangle = \langle x' + x'', u \rangle = \langle x', u \rangle + \langle x'', u \rangle = 0$. Since $x' \in \text{null } P$, $\langle x', u \rangle = 0$ (as $x' \in U^\perp$). Therefore we must have $\langle x'', u \rangle = 0$, implying that $x'' \in U^\perp$ also. Thus $x'' \in U$ and $x'' \in U^\perp$, but the only vector that can be in both a subspace and its orthogonal complement is 0, so $x'' = 0$, which contradicts our assumption above. □

From property 5 above, we must have $P_U^2 = P_U$, which implies that the eigenvalues of $P_U$ are 0 and 1. The eigenvectors belong to either the subspace $U$ (for eigenvalue 1) or to the orthogonal complement $U^\perp$ (for eigenvalue 0).

The orthonormal basis of a vector space $V$ can be divided into two separate lists of vectors, with one list $(e_1, \ldots, e_m)$ spanning the subspace $U$ and the other list $(f_1, \ldots, f_k)$ spanning $U^\perp$. A matrix representation of $P_U$ can be obtained by considering the action of $P_U$ on each of the basis vectors from the two subspaces. We have

$$P_U e_i = e_i \tag{6}$$
$$P_U f_i = 0 \tag{7}$$

In general, the matrix representation of an operator $T$ is defined in terms of its action on the basis vectors $v_i$ by

$$v'_j = \sum_{i=1}^{n} T_{ij} v_i \tag{8}$$

For a projection operator, we can see that this means that for the $m$ basis vectors $(e_1, \ldots, e_m)$ we must have $P_{ij} = \delta_{ij}$ for all $i, j = 1, \ldots, m$, while for the $k$ basis vectors $(f_1, \ldots, f_k)$ we must have $P_{ij} = 0$ for all $i, j = 1, \ldots, k$. If we list the basis vectors in the order $(e_1, \ldots, e_m, f_1, \ldots, f_k)$, then $P_U$ is a $(m+k) \times (m+k)$ diagonal matrix with the diagonal elements in the top $m$ rows equal to 1, and all other elements equal to zero.

In this basis, we see that det $P_U = 0$ (because there is at least one zero element on the diagonal) and tr $P_U = m$, which is the dimension of the subspace $U$. As the trace and determinant are invariant under a change of basis, these properties apply to any basis.

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