

UNITARY OPERATORS

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References: edX online course MIT 8.05.1x Week 4.

Sheldon Axler (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 7.

Another important type of operator is the *unitary operator* U , which is defined by the condition that it is surjective and that

$$|Uu| = |u| \quad (1)$$

for all $u \in V$. That is, a unitary operator preserves the norm of all vectors. The identity matrix I is a special case of a unitary operator, as it doesn't change any vector, but multiplying I by any complex number α with $|\alpha| = 1$ also preserves the norm, so αI is another unitary operator.

Because U preserves the norm of all vectors, the only vector that can be in the null space of U is the zero vector, meaning that U is also injective. As it is both injective and surjective, it is invertible.

Theorem 1. For a unitary operator U , $U^\dagger = U^{-1}$.

Proof. From its definition and the properties of an adjoint operator, we have

$$|Uu|^2 = \langle Uu, Uu \rangle \quad (2)$$

$$= \langle u, U^\dagger Uu \rangle \quad (3)$$

$$= \langle u, u \rangle \quad (4)$$

Therefore, $U^\dagger U = I$ so $U^\dagger = U^{-1}$. □

Theorem 2. Unitary operators preserve inner products, meaning that $\langle Uu, Uv \rangle = \langle u, v \rangle$ for all $u, v \in V$.

Proof. Since $U^\dagger = U^{-1}$ we have

$$\langle Uu, Uv \rangle = \langle u, U^\dagger Uv \rangle = \langle u, v \rangle \quad (5)$$

□

Theorem 3. Acting on an orthonormal basis (e_1, \dots, e_n) with a unitary operator U produces another orthonormal basis.

Proof. Suppose the orthonormal basis is converted to another set of vectors (f_1, \dots, f_n) by U :

$$f_i = U e_i \quad (6)$$

Then

$$\langle f_i, f_j \rangle = \langle U e_i, U e_j \rangle = \langle e_i, e_j \rangle = \delta_{ij} \quad (7)$$

Thus (f_1, \dots, f_n) are an orthonormal set. Since the orthonormal basis (e_1, \dots, e_n) spans V (by assumption) and the set (f_1, \dots, f_n) contains n linearly independent orthonormal vectors, (f_1, \dots, f_n) is also an orthonormal basis for V . \square

Theorem 4. *If one orthonormal basis (e_1, \dots, e_n) is converted to another (f_1, \dots, f_n) by a unitary operator U , then the matrix elements of U are the same in both bases.*

Proof. This is just a special case of the more general theorem that states that *any* operator that transforms one set of basis vectors into another has the same matrix elements in both bases. In this case, the proof is especially simple:

$$U_{ki}(\{e\}) = \langle e_k, U e_i \rangle \quad (8)$$

$$= \langle U^{-1} f_k, f_i \rangle \quad (9)$$

$$= \langle U^\dagger f_k, f_i \rangle \quad (10)$$

$$= \langle f_k, U f_i \rangle \quad (11)$$

$$= U_{ki}(\{f\}) \quad (12)$$

\square

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