NORMAL OPERATORS

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References: edX online course MIT 8.05 Week 6.

Hermitian and unitary operators are actually special cases of a more general type of operators known as *normal operators*. An operator is normal if it commutes with its adjoint:

$$\left[M^{\dagger},M\right]=0 \tag{1}$$

A hermitian operator is normal because $M^{\dagger}=M$. A unitary operator U is normal because $U^{\dagger}=U^{-1}$ and every operator commutes with its inverse.

Theorem 1. *A normal operator M remains normal after a similarity transformation with a unitary operator U.*

Proof. Suppose we transform M according to

$$M_{1}=U^{-1}MU=U^{\dagger}MU \tag{2}$$

Then

$$\begin{aligned}
\left[M_{1}^{\dagger},M_{1}\right] &= \left(U^{\dagger}M^{\dagger}U\right)\left(U^{\dagger}MU\right)-\left(U^{\dagger}MU\right)\left(U^{\dagger}M^{\dagger}U\right) && (3)\\
&= U^{\dagger}M^{\dagger}MU-U^{\dagger}MM^{\dagger}U && (4)\\
&= U^{\dagger}\left[M^{\dagger},M\right]U && (5)\\
&= 0 && (6)
\end{aligned}$$

Thus M_{1} is normal. □

Theorem 2. *If w is an eigenvector of the normal operator M with eigenvalue λ, it is also an eigenvector of the adjoint $M^{\dagger}$ with eigenvalue λ^{*}.*

Proof. Suppose

$$\begin{aligned}
Mw &= \lambda w && (7)\\
\left(M-\lambda I\right)w &= 0 && (8)
\end{aligned}$$

Consider the vector

$$u \equiv \left(M^{\dagger} - \lambda^{*} I\right) w \tag{9}$$

The norm of this vector satisfies

$$\langle u | u \rangle = \left\langle \left(M^{\dagger} - \lambda^{*} I\right) w \middle| \left(M^{\dagger} - \lambda^{*} I\right) w \right\rangle \tag{10}$$

$$= \left\langle w \middle| (M - \lambda I) \left(M^{\dagger} - \lambda^{*} I\right) w \right\rangle \tag{11}$$

$$= \left\langle w \middle| \left(M^{\dagger} - \lambda^{*} I\right) (M - \lambda I) w \right\rangle \tag{12}$$

$$= 0 \tag{13}$$

where in the third line we used 1 and in the last line we used 8. Since the norm of u is 0, $u = |0\rangle$ and since $w \neq 0$, from 9 we have

$$\left(M^{\dagger} - \lambda^{*} I\right) w = 0 \tag{14}$$

$$M^{\dagger} w = \lambda^{*} w \tag{15}$$

as required. □

PINGBACKS

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