NORMAL OPERATORS

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References: edX online course MIT 8.05 Week 6.

Hermitian and unitary operators are actually special cases of a more general type of operators known as *normal operators*. An operator is normal if it commutes with its adjoint:

$$\left[M^{\dagger},M\right]=0 \tag{1}$$

A hermitian operator is normal because $M^{\dagger}=M$. A unitary operator U is normal because $U^{\dagger}=U^{-1}$ and every operator commutes with its inverse.

Theorem 1. *A normal operator M remains normal after a similarity transformation with a unitary operator U.*

Proof. Suppose we transform M according to

$$M_{1}=U^{-1}MU=U^{\dagger}MU \tag{2}$$

Then

$$\begin{aligned}
\left[M_{1}^{\dagger},M_{1}\right] &= \left(U^{\dagger}M^{\dagger}U\right)\left(U^{\dagger}MU\right)-\left(U^{\dagger}MU\right)\left(U^{\dagger}M^{\dagger}U\right) && (3)\\
&= U^{\dagger}M^{\dagger}MU-U^{\dagger}MM^{\dagger}U && (4)\\
&= U^{\dagger}\left[M^{\dagger},M\right]U && (5)\\
&= 0 && (6)
\end{aligned}$$

Thus M_{1} is normal. □

Theorem 2. *If w is an eigenvector of the normal operator M with eigenvalue λ, it is also an eigenvector of the adjoint $M^{\dagger}$ with eigenvalue λ^{*}.*

Proof. Suppose

$$\begin{aligned}
Mw &= \lambda w && (7)\\
(M-\lambda I)\,w &= 0 && (8)
\end{aligned}$$

Consider the vector

$$u \equiv \left(M^{\dagger}-\lambda^{*}I\right)w \tag{9}$$

The norm of this vector satisfies

$$\begin{aligned}
\langle u|u\rangle &= \left\langle \left(M^{\dagger}-\lambda^{*}I\right)w \middle| \left(M^{\dagger}-\lambda^{*}I\right)w \right\rangle && (10)\\
&= \left\langle w \middle| (M-\lambda I)\left(M^{\dagger}-\lambda^{*}I\right)w \right\rangle && (11)\\
&= \left\langle w \middle| \left(M^{\dagger}-\lambda^{*}I\right)(M-\lambda I)w \right\rangle && (12)\\
&= 0 && (13)
\end{aligned}$$

where in the third line we used 1 and in the last line we used 8. Since the norm of u is 0, $u=|0\rangle$ and since $w\neq 0$, from 9 we have

$$\begin{aligned}
\left(M^{\dagger}-\lambda^{*}I\right)w &= 0 && (14)\\
M^{\dagger}w &= \lambda^{*}w && (15)
\end{aligned}$$

as required. □

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