

NORMAL OPERATORS

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References: edX online course MIT 8.05 Week 6.

Hermitian and unitary operators are actually special cases of a more general type of operators known as *normal operators*. An operator is normal if it commutes with its adjoint:

$$(0.1) \quad [M^\dagger, M] = 0$$

A hermitian operator is normal because $M^\dagger = M$. A unitary operator U is normal because $U^\dagger = U^{-1}$ and every operator commutes with its inverse.

Theorem 1. *A normal operator M remains normal after a similarity transformation with a unitary operator U .*

Proof. Suppose we transform M according to

$$(0.2) \quad M_1 = U^{-1}MU = U^\dagger MU$$

Then

$$(0.3) \quad [M_1^\dagger, M_1] = (U^\dagger M^\dagger U) (U^\dagger MU) - (U^\dagger MU) (U^\dagger M^\dagger U)$$

$$(0.4) \quad = U^\dagger M^\dagger MU - U^\dagger MM^\dagger U$$

$$(0.5) \quad = U^\dagger [M^\dagger, M] U$$

$$(0.6) \quad = 0$$

Thus M_1 is normal. □

Theorem 2. *If w is an eigenvector of the normal operator M with eigenvalue λ , it is also an eigenvector of the adjoint M^\dagger with eigenvalue λ^* .*

Proof. Suppose

$$(0.7) \quad Mw = \lambda w$$

$$(0.8) \quad (M - \lambda I)w = 0$$

Consider the vector

$$(0.9) \quad u \equiv (M^\dagger - \lambda^* I) w$$

The norm of this vector satisfies

$$(0.10) \quad \langle u|u \rangle = \langle (M^\dagger - \lambda^* I) w | (M^\dagger - \lambda^* I) w \rangle$$

$$(0.11) \quad = \langle w | (M - \lambda I) (M^\dagger - \lambda^* I) w \rangle$$

$$(0.12) \quad = \langle w | (M^\dagger - \lambda^* I) (M - \lambda I) w \rangle$$

$$(0.13) \quad = 0$$

where in the third line we used 0.1 and in the last line we used 0.8. Since the norm of u is 0, $u = |0\rangle$ and since $w \neq 0$, from 0.9 we have

$$(0.14) \quad (M^\dagger - \lambda^* I) w = 0$$

$$(0.15) \quad M^\dagger w = \lambda^* w$$

as required. □

PINGBACKS

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