VECTOR SPACES & LINEAR INDEPENDENCE - SOME EXAMPLES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.1.1 - 1.1.5.

Here are a few examples of vector space problems.

Given the axioms of a vector space, we can derive a few more properties. I'll use Shankar's notation for vectors, which is essentially Dirac's bra-ket notation.

Theorem 1. *The additive identity* 0 *is unique.*

Proof. Proof: (by contradiction). Suppose there are two distinct additive identities $|0\rangle$ and $|0'\rangle$. Then

$$\left|0'\right\rangle = \left|0'\right\rangle + |0\rangle \text{ (since } |0\rangle \text{is an additive identity)} \tag{0.1}$$

$$= |0\rangle + \left|0'\right\rangle \text{ (commutative addition)} \tag{0.2}$$

$$= |0\rangle \text{ (since } |0'\rangle \text{is an additive identity)} \tag{0.3}$$

□

Theorem 2. *Multiplication of any vector by the zero scalar gives the zero vector.*

Proof. We wish to show that $0|v\rangle = |0\rangle$ for all $v \in V$. We have

$$|0\rangle = (0+1)|v\rangle + |-v\rangle \tag{0.4}$$

$$= 0|v\rangle + |v\rangle + |-v\rangle \tag{0.5}$$

$$= 0|v\rangle + |0\rangle \tag{0.6}$$

$$= 0|v\rangle \tag{0.7}$$

where the third line follows because $|-v\rangle$ is the additive inverse of $|v\rangle$ and the last line follows because $|0\rangle$ is the additive identity vector. □

Theorem 3. $|-v\rangle = -|v\rangle$. *That is,* $-|v\rangle$ *is the additive inverse of* $|v\rangle$.

Proof. The negative of a vector v is multiplication of v by the scalar -1, so

$$|v\rangle + (-|v\rangle) = (1 + (-1))\,|v\rangle \tag{0.8}$$
$$= 0\,|v\rangle \tag{0.9}$$
$$= |0\rangle \tag{0.10}$$

by theorem 2. Thus $-|v\rangle$ is an additive inverse of $|v\rangle$, so $-|v\rangle = |-v\rangle$. □

Theorem 4. *The additive inverse $|-v\rangle$ is unique.*

Proof. Suppose there is another vector $|w\rangle$ for which $|v\rangle + |w\rangle = |0\rangle$. By theorem 1, $|0\rangle$ is unique, so we must have $|v\rangle + |w\rangle = |v\rangle + |-v\rangle$. By theorem 3, this gives

$$|v\rangle - |v\rangle + |w\rangle = |-v\rangle \tag{0.11}$$
$$|0\rangle + |w\rangle = |-v\rangle \tag{0.12}$$
$$|w\rangle = |-v\rangle \tag{0.13}$$

where the third line follows because $|0\rangle$ is the additive identity. □

Example 5. Consider the set of all entities (a,b,c) where the entries are real numbers. Addition and scalar multiplication are defined as

$$(a,b,c) + (d,e,f) \equiv (a+d, b+e, c+f) \tag{0.14}$$
$$\alpha\,(a,b,c) \equiv (\alpha a, \alpha b, \alpha c) \tag{0.15}$$

The null vector is

$$|0\rangle = (0,0,0) \tag{0.16}$$

The inverse of (a,b,c) is $(-a,-b,-c)$. As the set is closed under addition and scalar multiplication it is a vector space. However, a subset such as $(a,b,1)$ is *not* a vector space since it is not closed under addition or scalar multiplication:

$$(a,b,1) + (d,e,1) = (a+d, b+e, 2) \tag{0.17}$$
$$2\,(a,b,1) = (2a, 2b, 2) \tag{0.18}$$

Neither of the vectors on the RHS are of the form $(a,b,1)$ so they don't lie in the set.

Example 6. The set of all functions $f(x)$ defined on an interval $0 \leq x \leq L$ form a vector space if we define addition as pointwise addition $f+g = f(x)+g(x)$ for all x, and scalar multiplication by a as $af(x)$.

Some subsets of this vector space are also vector spaces. For example the set of all functions that satsify $f(0) = f(L) = 0$ is a vector space, because the sum of any two such functions also satisfies $(f+g)(0) = (f+g)(L) = 0$, and scalar multiplication leaves the endpoints at 0 as well.

The subset of periodic functions $f(0) = f(L)$ (not necessarily equal to 0) is also a vector space. Adding any two functions from this subset gives a sum such that

$$f(0)+g(0) = f(L)+g(L) \tag{0.19}$$
$$(f+g)(0) = (f+g)(L) \tag{0.20}$$

Multiplying by a scalar gives

$$a(f(0)+g(0)) = a(f(L)+g(L)) \tag{0.21}$$
$$a(f+g)(0) = a(f+g)(L) \tag{0.22}$$

However, a subset such as all functions with $f(0) = 4$ is not a vector space, since adding two such functions gives a sum with $(f+g)(0) = 8$, and multiplying by a scalar gives a function with $af(0) = 4a$, neither of which is in the subset.

Now a couple of examples of linear independence.

Example 7. We have three vectors from the vector space of real 2×2 matrices:

$$|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{0.23}$$
$$|2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{0.24}$$
$$|3\rangle = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} \tag{0.25}$$

These are not linearly independent, because $|3\rangle = |1\rangle - 2|2\rangle$.

Example 8. We have 3 row vectors

$$|1\rangle = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \tag{0.26}$$
$$|2\rangle = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \tag{0.27}$$
$$|3\rangle = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \tag{0.28}$$

These are linearly dependent, since $|3\rangle = 2|1\rangle + |2\rangle$.
Now we look at the 3 vectors

$$|1\rangle = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \tag{0.29}$$
$$|2\rangle = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \tag{0.30}$$
$$|3\rangle = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \tag{0.31}$$

We can show that these are linearly independent by attempting to solve the equation

$$0 = a|1\rangle + b|2\rangle + c|3\rangle \tag{0.32}$$

Looking at each component, we have

$$a + b = 0 \tag{0.33}$$
$$a + c = 0 \tag{0.34}$$
$$b + c = 0 \tag{0.35}$$

Solving the last two equations for a and b in terms of c and substituting into the first equation, we get

$$-2c = 0 \tag{0.36}$$
$$c = 0 \tag{0.37}$$

Thus we find that the only solution is $a = b = c = 0$, which proves linear independence.

PINGBACKS

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