

VECTOR SPACES & LINEAR INDEPENDENCE - SOME EXAMPLES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.1.1 - 1.1.5.

Here are a few examples of vector space problems.

Given the axioms of a vector space, we can derive a few more properties. I'll use Shankar's notation for vectors, which is essentially Dirac's bra-ket notation.

Theorem 1. *The additive identity 0 is unique.*

Proof. Proof: (by contradiction). Suppose there are two distinct additive identities $|0\rangle$ and $|0'\rangle$. Then

$$|0'\rangle = |0'\rangle + |0\rangle \text{ (since } |0\rangle \text{ is an additive identity)} \quad (1)$$

$$= |0\rangle + |0'\rangle \text{ (commutative addition)} \quad (2)$$

$$= |0\rangle \text{ (since } |0'\rangle \text{ is an additive identity)} \quad (3)$$

□

Theorem 2. *Multiplication of any vector by the zero scalar gives the zero vector.*

Proof. We wish to show that $0|v\rangle = |0\rangle$ for all $v \in V$. We have

$$|0\rangle = (0 + 1)|v\rangle + |-v\rangle \quad (4)$$

$$= 0|v\rangle + |v\rangle + |-v\rangle \quad (5)$$

$$= 0|v\rangle + |0\rangle \quad (6)$$

$$= 0|v\rangle \quad (7)$$

where the third line follows because $|-v\rangle$ is the additive inverse of $|v\rangle$ and the last line follows because $|0\rangle$ is the additive identity vector. □

Theorem 3. $|-v\rangle = -|v\rangle$. That is, $-|v\rangle$ is the additive inverse of $|v\rangle$.

Proof. The negative of a vector v is multiplication of v by the scalar -1 , so

$$|v\rangle + (-|v\rangle) = (1 + (-1))|v\rangle \quad (8)$$

$$= 0|v\rangle \quad (9)$$

$$= |0\rangle \quad (10)$$

by theorem 2. Thus $-|v\rangle$ is an additive inverse of $|v\rangle$, so $-|v\rangle = |-v\rangle$. \square

Theorem 4. *The additive inverse $|-v\rangle$ is unique.*

Proof. Suppose there is another vector $|w\rangle$ for which $|v\rangle + |w\rangle = |0\rangle$. By theorem 1, $|0\rangle$ is unique, so we must have $|v\rangle + |w\rangle = |v\rangle + |-v\rangle$. By theorem 3, this gives

$$|v\rangle - |v\rangle + |w\rangle = |-v\rangle \quad (11)$$

$$|0\rangle + |w\rangle = |-v\rangle \quad (12)$$

$$|w\rangle = |-v\rangle \quad (13)$$

where the third line follows because $|0\rangle$ is the additive identity. \square

Example 5. Consider the set of all entities (a, b, c) where the entries are real numbers. Addition and scalar multiplication are defined as

$$(a, b, c) + (d, e, f) \equiv (a + d, b + e, c + f) \quad (14)$$

$$\alpha(a, b, c) \equiv (\alpha a, \alpha b, \alpha c) \quad (15)$$

The null vector is

$$|0\rangle = (0, 0, 0) \quad (16)$$

The inverse of (a, b, c) is $(-a, -b, -c)$. As the set is closed under addition and scalar multiplication it is a vector space. However, a subset such as $(a, b, 1)$ is *not* a vector space since it is not closed under addition or scalar multiplication:

$$(a, b, 1) + (d, e, 1) = (a + d, b + e, 2) \quad (17)$$

$$2(a, b, 1) = (2a, 2b, 2) \quad (18)$$

Neither of the vectors on the RHS are of the form $(a, b, 1)$ so they don't lie in the set.

Example 6. The set of all functions $f(x)$ defined on an interval $0 \leq x \leq L$ form a vector space if we define addition as pointwise addition $f + g = f(x) + g(x)$ for all x , and scalar multiplication by a as $af(x)$.

Some subsets of this vector space are also vector spaces. For example the set of all functions that satisfy $f(0) = f(L) = 0$ is a vector space, because the sum of any two such functions also satisfies $(f+g)(0) = (f+g)(L) = 0$, and scalar multiplication leaves the endpoints at 0 as well.

The subset of periodic functions $f(0) = f(L)$ (not necessarily equal to 0) is also a vector space. Adding any two functions from this subset gives a sum such that

$$f(0) + g(0) = f(L) + g(L) \quad (19)$$

$$(f+g)(0) = (f+g)(L) \quad (20)$$

Multiplying by a scalar gives

$$a(f(0) + g(0)) = a(f(L) + g(L)) \quad (21)$$

$$a(f+g)(0) = a(f+g)(L) \quad (22)$$

However, a subset such as all functions with $f(0) = 4$ is not a vector space, since adding two such functions gives a sum with $(f+g)(0) = 8$, and multiplying by a scalar gives a function with $af(0) = 4a$, neither of which is in the subset.

Now a couple of examples of linear independence.

Example 7. We have three vectors from the vector space of real 2×2 matrices:

$$|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (23)$$

$$|2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (24)$$

$$|3\rangle = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} \quad (25)$$

These are not linearly independent, because $|3\rangle = |1\rangle - 2|2\rangle$.

Example 8. We have 3 row vectors

$$|1\rangle = [1 \ 1 \ 0] \quad (26)$$

$$|2\rangle = [1 \ 0 \ 1] \quad (27)$$

$$|3\rangle = [3 \ 2 \ 1] \quad (28)$$

These are linearly dependent, since $|3\rangle = 2|1\rangle + |2\rangle$.

Now we look at the 3 vectors

$$|1\rangle = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \quad (29)$$

$$|2\rangle = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \quad (30)$$

$$|3\rangle = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \quad (31)$$

We can show that these are linearly independent by attempting to solve the equation

$$0 = a|1\rangle + b|2\rangle + c|3\rangle \quad (32)$$

Looking at each component, we have

$$a + b = 0 \quad (33)$$

$$a + c = 0 \quad (34)$$

$$b + c = 0 \quad (35)$$

Solving the last two equations for a and b in terms of c and substituting into the first equation, we get

$$-2c = 0 \quad (36)$$

$$c = 0 \quad (37)$$

Thus we find that the only solution is $a = b = c = 0$, which proves linear independence.

PINGBACKS

Pingback: Vector spaces: span, linear independence and basis