# GRAM-SCHMIDT ORTHOGONALIZATION - A COUPLE OF EXAMPLES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.3.1 - 1.3.2.

Here are a couple of examples of the Gram-Schmidt orthogonalization procedure. The recipe for generating an orthonormal basis $e_i$ from a general set of linearly independent vectors $v_i$ is as follows.

The first vector $e_1$ in the orthonormal basis is defined by

$$e_1 = \frac{v_1}{|v_1|} \tag{1}$$

where $v_1$ is the first vector (well, any vector, really) in the non-orthonormal basis.

Given vector $e_{j-1}$ in the orthonormal basis, we can form $e_j$ from the formula

$$e_j = \frac{v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \tag{2}$$

**Example 1.** Given $v_1 = (3,4)$ and $v_2 = (2,-6)$ we can form an orthonormal basis in two ways. Starting with $v_1$ we have

$$e_1 = \frac{v_1}{|v_1|} = \left( \frac{3}{5}, \frac{4}{5} \right) \tag{3}$$

$$e_2 = \frac{v_2 - \langle e_1, v_2 \rangle e_1}{|v_2 - \langle e_1, v_2 \rangle e_1|} \tag{4}$$

To evaluate $e_2$, we have

$$\langle e_1, v_2 \rangle = \frac{6}{5} - \frac{24}{5} = -\frac{18}{5} \tag{5}$$

$$v_2 - \langle e_1, v_2 \rangle e_1 = (2, -6) + \frac{18}{5}\left(\frac{3}{5}, \frac{4}{5}\right) \tag{6}$$

$$= \frac{1}{25}(104, -78) \tag{7}$$

$$|v_2 - \langle e_1, v_2 \rangle e_1| = \frac{130}{25} \tag{8}$$

$$e_2 = \frac{1}{130}(104, -78) \tag{9}$$

As a check,

$$\langle e_1, e_2 \rangle = \frac{1}{650}(312 - 312) = 0 \tag{10}$$

$$\langle e_1, e_1 \rangle = \frac{1}{25}(9 + 16) = 1 \tag{11}$$

$$\langle e_2, e_2 \rangle = \frac{1}{16900}(10816 + 6084) = 1 \tag{12}$$

We could also start with $v_2$, giving

$$e_1 = \frac{v_2}{|v_2|} = \frac{1}{2\sqrt{10}}(2, -6) \tag{13}$$

$$e_2 = \frac{v_1 - \langle e_1, v_1 \rangle e_1}{|v_1 - \langle e_1, v_1 \rangle e_1|} \tag{14}$$

$$\langle e_1, v_1 \rangle = \frac{1}{2\sqrt{10}}(6 - 24) = -\frac{9}{\sqrt{10}} \tag{15}$$

$$v_1 - \langle e_1, v_1 \rangle e_1 = (3, 4) + \frac{9}{20}(2, -6) \tag{16}$$

$$= \frac{1}{20}(78, 26) \tag{17}$$

$$|v_1 - \langle e_1, v_1 \rangle e_1| = \frac{\sqrt{6760}}{20} \tag{18}$$

$$e_2 = \frac{1}{\sqrt{6760}}(78, 26) \tag{19}$$

Checking, we get

$$\langle e_1, e_2 \rangle = \frac{1}{2\sqrt{67600}}(156-156) = 0 \tag{20}$$

$$\langle e_1, e_1 \rangle = \frac{1}{40}(4+36) = 1 \tag{21}$$

$$\langle e_2, e_2 \rangle = \frac{1}{6760}(6084+676) = 1 \tag{22}$$

**Example 2.** We're now given 3 vectors in 3-d space:

$$v_1 = (3,0,0) \tag{23}$$

$$v_2 = (0,1,2) \tag{24}$$

$$v_3 = (0,2,5) \tag{25}$$

The problem is to generate linear combinations of these 3 vectors to give the orthonormal basis

$$e_1 = (1,0,0) \tag{26}$$

$$e_2 = \frac{1}{\sqrt{5}}(0,1,2) \tag{27}$$

$$e_3 = \frac{1}{\sqrt{5}}(0,-2,1) \tag{28}$$

We could use the Gram-Schmidt procedure, but it's probably easier to just solve the equations. We have

$$e_1 = \frac{v_1}{3} \tag{29}$$

$$e_2 = \frac{v_2}{\sqrt{5}} \tag{30}$$

$$e_3 = Av_1 + Bv_2 + Cv_3 \tag{31}$$

Writing out the last equation using components, we have

$$0 = A \tag{32}$$

$$-\frac{2}{\sqrt{5}} = B + 2C \tag{33}$$

$$\frac{1}{\sqrt{5}} = 2B + 5C \tag{34}$$

The solution is

$$C = \frac{5}{\sqrt{5}} = \sqrt{5} \tag{35}$$

$$B = -\frac{12}{\sqrt{5}} \tag{36}$$

Thus we have

$$e_1 = \frac{v_1}{3} \tag{37}$$

$$e_2 = \frac{v_2}{\sqrt{5}} \tag{38}$$

$$e_3 = -\frac{12}{\sqrt{5}} v_2 + \sqrt{5} v_3 \tag{39}$$