TRIANGLE INEQUALITY AS AN EQUALITY

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We’ve already proved the triangle inequality for vectors, but it’s worth adding a note on when the inequality becomes an equality. The triangle inequality states that for all \( u, v \in V \)

\[
|u + v| \leq |u| + |v|
\]  

(1)

To make this an equality, we need to look back at the proof. The last step in the proof invokes the Schwarz inequality to state that

\[
|u + v|^2 \leq |u|^2 + |v|^2 + 2|u||v|
\]

(2)

Looking at the proof for the Schwarz inequality, we see that it becomes an equality if the component \( w \) of \( u \) that is orthogonal to \( v \) is zero, that is, if \( u = \alpha v \) for some (possibly complex) scalar \( \alpha \). If that is the case, then

\[
|u + v| = |\alpha v + v| = |1 + \alpha||v|
\]

(3)

\[
|u| + |v| = |\alpha v| + |v| = (1 + |\alpha|)|v|
\]

(4)

Thus the triangle inequality becomes an equality if

\[
|1 + \alpha| = 1 + |\alpha|
\]

(5)

which occurs if \( \alpha \) is real and \( \alpha \geq 0 \). In terms of vectors as arrows in 3-d space, this condition is equivalent to the two vectors being parallel and pointing in the same direction (rather than in opposite directions).

To see that equality doesn’t happen if \( \alpha \) is complex, suppose \( \alpha = 1 + i \). Then

\[
|1 + \alpha| = |2 + i| = \sqrt{5}
\]

(6)

\[
1 + |\alpha| = 1 + \sqrt{2}
\]

(7)