

UNITARY MATRICES - SOME EXAMPLES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.6.3 - 1.6.6.

Here are a few more results about unitary operators.

Shankar defines a unitary operator U as one where

$$(0.1) \quad UU^\dagger = I$$

From this we can derive the other condition by which they can be defined, namely that a unitary operator preserves the norm of a vector:

$$(0.2) \quad |Uv| = |v|$$

This follows, for if we define the effect of U by

$$(0.3) \quad |v'_1\rangle = U|v_1\rangle$$

then

$$(0.4) \quad \langle v'_1 | v'_1 \rangle = \langle Uv_1 | Uv_1 \rangle$$

$$(0.5) \quad = \langle v_1 | U^\dagger U v_1 \rangle$$

$$(0.6) \quad = \langle v_1 | v_1 \rangle$$

Thus $|v'_1|^2 = |v_1|^2$.

Theorem 1. *The product of two unitary operators U_1 and U_2 is unitary.*

Proof. Using Shankar's definition 0.1, we have

$$(0.7) \quad (U_1 U_2)^\dagger U_1 U_2 = U_2^\dagger U_1^\dagger U_1 U_2$$

$$(0.8) \quad = U_2^\dagger I U_2$$

$$(0.9) \quad = U_2^\dagger U_2$$

$$(0.10) \quad = I$$

□

Theorem 2. *The determinant of a unitary matrix U is a complex number with unit modulus.*

Proof. The determinant of a hermitian conjugate is the complex conjugate of the determinant of the original matrix, since $\det U = \det U^T$ (where the superscript T denotes the transpose) for any matrix, and the hermitian conjugate is the complex conjugate transpose. Therefore

$$(0.11) \quad \det(UU^\dagger) = [\det U][\det U]^* = \det I = 1$$

Therefore $|\det U|^2 = 1$ as required. \square

Example 3. The rotation matrix $R\left(\frac{\pi}{2}\mathbf{i}\right)$ is unitary. We have

$$(0.12) \quad R\left(\frac{\pi}{2}\mathbf{i}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

By direct calculation

$$(0.13) \quad RR^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$(0.14) \quad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Example 4. Consider the matrix

$$(0.15) \quad U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

By calculating

$$(0.16) \quad UU^\dagger = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

$$(0.17) \quad = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I$$

Thus U is unitary, but because $U \neq U^\dagger$ it is not hermitian. Its determinant is

$$(0.18) \quad \det U = \left(\frac{1}{\sqrt{2}} \right)^2 (1 - i^2) = 1$$

This is of the required form $e^{i\theta}$ with $\theta = 0$.

Example 5. Consider the matrix

$$(0.19) \quad U = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

$$(0.20) \quad UU^\dagger = \frac{1}{4} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$(0.21) \quad = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I$$

Thus U is unitary, but because $U \neq U^\dagger$ it is not hermitian. Its determinant is

$$(0.22) \quad \det U = \left(\frac{1}{2} \right)^2 [(1+i)^2 - (1-i)^2]$$

$$(0.23) \quad = i$$

This is of the required form $e^{i\theta}$ with $\theta = \frac{\pi}{2}$.

PINGBACKS

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